REPUBLIQUE ALGERIENNE DEMOCRATIQUE ET POPULAIRE
Ministère de l'Enseignement supérieur
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École Supérieure de Commerce

Course Handout
PORTFOLIO THEORY
Lectures and Tutorials

Level: 2nd Year Master's in Corporate Finance
Total Hours: 45 Hours

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Course Overview

This Master's-level course provides a rigorous exploration of the theoretical underpinnings and empirical realities of modern investment analysis, beginning with the foundational principles of Modern Portfolio Theory (MPT). Students will delve deeply into the Mean-Variance Framework, mastering the quantitative techniques for constructing optimal portfolios that seek to maximize expected return for a given level of risk. Building directly upon this foundation, the course critically examines the Capital Asset Pricing Model (CAPM), analyzing its derivation, its profound implications for the relationship between expected return and systematic risk, and its role as a cornerstone of asset pricing. A significant component of the curriculum is dedicated to the empirical validation of these seminal models, where students will engage with the body of scholarly evidence that both supports and challenges their predictions, fostering a sophisticated understanding of their practical applications and limitations in real-world financial markets.

Key Topics Covered

Introduction

- The Mean/Variance approach.
- Definition of the portfolio problem.

1. **Normative Models of Financial Asset Selection**
   (Modèles normatifs de choix des actifs financiers)
   - Portfolio return and risk (Rendement et risque d’un portefeuille)
   - Two-asset and multi-asset portfolio selection (Sélection de portefeuilles à 2 titres et à n titres)
   - Minimum variance portfolio (Portefeuille de variance minimale)
   - Efficient frontier and efficient set (Frontière efficiente et ensemble efficient)
   - Risk-return trade-off in efficient portfolios (Relation rendement-risque des portefeuilles efficients)

2. **Advanced Portfolio Models & Market Equilibrium**
   (Modèles avancés et équilibre du marché financier)

- Introduction of the risk-free asset and the Separation Theorem (Actif sans risque et théorème de séparation)
- Capital Market Line (CML) (Ligne de marché des capitaux)
- Diagonal model and Market model (Modèle diagonal et modèle de marché)
- Benefits of diversification (Gain de la diversification)
- Short-sale constraints and numerical solutions (Contraintes sur les ventes à découvert)

3. **The Capital Asset Pricing Model (CAPM)**
   (Modèle d'évaluation des actifs financiers)ù
   - CAPM assumptions (Hypothèses du CAPM)
   - Derivation and interpretation (Démonstration et interprétation)
   - Security Market Line (SML) (Ligne de marché des titres)

4. **Empirical Validation of CAPM & Extensions**
   (Validation empirique et extensions du CAPM)
   - Market model vs. CAPM (Modèle de marché vs. CAPM)
   - Beta estimation (Estimation des betas)
   - Fama & French multi-factor model (Modèle multi-facteurs)

5. **Applications & Extensions**
   (Applications et développements récents)
   - Performance evaluation of mutual funds (Mesures de performance des fonds d'investissement)
   - Skewness and three-moment portfolio models (Asymétrie et modèles à trois paramètres)
   - Arbitrage Pricing Theory (APT) (Théorie d'évaluation par arbitrage)

Learning Objectives

By the end of this course, students will:

- Understand the mathematical foundations of portfolio optimization.
- Apply the CAPM and APT to asset pricing problems.
- Evaluate portfolio performance using risk-adjusted metrics.

- Critically assess empirical challenges to traditional asset pricing models.

This course combines theoretical rigor with practical applications, preparing students for careers in asset management, risk analysis, and financial research.

Introduction

This course offers a comprehensive exploration of the fundamental principles governing portfolio construction, asset valuation, and investment strategies in financial markets. It bridges rigorous theoretical frameworks with practical applications to equip students with the analytical tools needed for informed decision-making in real-world financial environments.

We begin by examining the foundational concepts of risk-return trade-offs, emphasizing diversification strategies and optimization techniques that minimize risk without compromising expected returns. Students will learn how to construct efficient portfolios by analyzing asset correlations, volatility, and market constraints, gaining insights into how rational investors allocate capital across different securities.

The course then delves into equilibrium-based asset pricing models, with a focus on the Capital Asset Pricing Model (CAPM) and its implications for expected returns relative to systematic risk. We will critically assess the model's assumptions, empirical validity, and limitations before exploring modern extensions such as the Arbitrage Pricing Theory (APT) and multi-factor models that account for anomalies beyond traditional market risk.

Finally, the course covers advanced topics in performance evaluation, including risk-adjusted metrics and behavioral biases that influence investment decisions. By integrating theoretical knowledge with empirical analysis, students will develop a nuanced understanding of portfolio management, asset pricing dynamics, and the challenges of achieving consistent returns in evolving financial markets. This course is designed for future investment professionals seeking a balanced perspective on both classical theories and contemporary developments in finance.

1. Introduction to Mean-Variance Approach

1.1 Mean-Variance Portfolio Theory is a fundamental framework in financial economics that provides a quantitative approach to portfolio selection. Developed by (Markowitz, 1952), it forms the basis of Modern Portfolio Theory (MPT). The theory assumes that investors are rational and risk-averse, seeking to maximize expected return for a given level of risk or minimize risk for a given level of expected return. It relies on the statistical properties of asset returns, specifically their expected returns (mean) and risk (variance), to construct optimal portfolios.

Formal Definition: Mean-variance portfolio theory is a mathematical framework for constructing portfolios that optimize the trade-off between expected return and risk. It involves selecting asset weights in a portfolio to achieve one of two objectives:

1. **Maximizing expected portfolio return** for a given level of portfolio risk (variance).
2. **Minimizing portfolio risk** (variance) for a given level of expected return.

The theory is based on the following key components:

1. **Expected Return**: The weighted average of the expected returns of individual assets in the portfolio.
2. **Portfolio Variance**: A measure of portfolio risk, calculated as the weighted sum of the variances and covariances of the individual asset returns.
3. **Efficient Frontier**: The set of portfolios that offer the highest expected return for each level of risk or the lowest risk for each level of expected return.
4. **Optimization**: The process of solving for the asset weights that achieve the desired risk-return trade-off, typically using quadratic programming.

1.2 Assumptions:

1. Investors are rational and risk-averse.
2. Asset returns are normally distributed or can be fully described by their mean and variance.
3. Markets are efficient, and all relevant information is reflected in asset prices.
4. Investors have a single-period investment horizon.

2. Key Concepts

- **Expected Return**: The average return an investor anticipates from an investment.

$$E(R)_P = \sum_{n=1}^{N} w_n E(R)_n$$

- **Risk (Variance)**: Measures the variability of returns.

$$\sigma_P^2 = \sum_{j=1}^{N} w_j^2 \sigma_j^2 + \left[\sum_{j=1}^{N} \sum_{k=1, k \neq j}^{N} w_j w_k \sigma_{jk} \right]$$

- **Covariance**: Measures how two assets move together.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

2. Definition of the Portfolio Problem

The portfolio problem is a fundamental challenge in financial theory that involves determining the optimal allocation of investments among various assets to achieve the best possible trade-off between expected return and risk. Formally, it requires selecting asset weights that either maximize expected portfolio return for a given level of risk (measured by variance or standard deviation) or minimize risk for a target expected return, while satisfying constraints such as budget limitations (sum of weights equals 1) and potentially no short-selling restrictions.

This optimization framework, pioneered by Harry Markowitz's Modern Portfolio Theory (MPT), relies on statistical measures of asset returns (means, variances, and covariances) and generates the efficient frontier—the set of portfolios offering maximum return for each risk level. The problem's solution depends critically on the covariance structure between assets, as diversification benefits emerge when combining assets with less-than-perfect positive correlations. The inclusion of a risk-free asset further extends the solution to the capital market line (CML), representing optimal combinations of risky and risk-free investments. The portfolio problem thus provides the mathematical foundation for systematic asset allocation and remains central to both academic finance and practical investment management.

2.1. The Concept of Risk Aversion

Risk aversion describes the fundamental preference of most investors for certainty over uncertainty, leading them to prefer less risk to more risk, all else being equal. An investor's risk tolerance quantifies the amount of risk they are willing to accept in pursuit of higher returns; it is the inverse of risk aversion, meaning high risk aversion is synonymous with low risk tolerance.

Investors can be broadly categorized into three behavioral profiles based on their attitude toward risk and return:

- **Risk-Averse:** This is the predominant profile. A risk-averse investor, when faced with two investments offering the same expected return but different levels of risk, will always prefer the less risky alternative. They require compensation in the form of higher expected returns to accept additional risk.
- **Risk-Neutral:** A risk-neutral investor is indifferent to risk. Their decisions are based solely on the magnitude of expected return, with the level of risk having no bearing on their utility.
- **Risk-Seeking:** In contrast to the risk-averse, a risk-seeking investor derives utility from risk itself. When presented with two investments with identical expected returns, they will actively prefer the one with higher risk.

Chapter 1: Normative Models of Financial Asset Selection

Normative models of financial asset selection provide a structured, theory-driven approach to choosing investments based on rational decision-making principles. Unlike descriptive models, which explain how investors *actually* behave, normative models prescribe how investors *should* behave to achieve optimal outcomes, such as maximizing returns for a given level of risk. These models rely on mathematical and economic theories to guide portfolio construction, asset allocation, and risk management.

1. **Measuring risk and return**

1.1. Measuring risk and return for a single asset

1.1.1 Measures of Location

Measures of location describe the central tendency or "average" of a dataset. They help summarize the typical return of an asset.

a. Mean (Arithmetic Mean)

- **Definition**: The average return of an asset over a period.
- **Formula**: Mean = (Sum of Returns) / (Number of Returns).
- **Example**: For returns 5%, 10%, -2%, and 7%, the mean is (5 + 10 - 2 + 7) / 4 =5%.

b. Median

- **Definition**: The middle value of a dataset when returns are arranged in ascending or descending order.
- **Steps**:
    1. Sort the returns.
    2. If n is odd, the median is the middle value.
    3. If n is even, the median is the average of the two middle values.
- **Example**: For returns $\{5\%,10\%,-2\%,7\%\}\{5\%,10\%,-2\%,7\%\}$:
    1. Sorted returns: $\{-2\%,5\%,7\%,10\%\}\{-2\%,5\%,7\%,10\%\}$.
    2. Median = 5%+7%/2=6%.

c. Mode

- **Definition**: The most frequently occurring return.

- **Example**: For returns 5%, 10%, -2%, 7%, 5%, the mode is 5%.

Several parameters can be used: mean, mode and median. The second is not appropriate for prices and rates (continuous variables) and the third cannot be used with skewed distributions.

d. Expected Return

- **Definition**: The weighted average of possible returns, based on their probabilities.
- **Formula**:

$$E(\hat{P}) = \sum_{i=1}^{n} P_i P(P_i)$$

- **Example**: For returns 10% (30% probability), 5% (50% probability), and -2% (20% probaility), the expected return is :

1.1.2. Measures of Dispersion

Measures of dispersion describe the variability or risk of an asset's returns. They quantify how much the returns deviate from the mean.

a. Variance

- **Definition**: The average squared deviation of returns from the mean.
- **Formula**:

$$V(\hat{P}) = \sum_{i=1}^{n} (P_i - E(\hat{P}))^2 P(P_i)$$

Example: For returns {5%,10%,−2%,7%}{5%,10%,−2%,7%} with a mean of 5%:

σ^2=19.5

b. Standard Deviation

- **Definition**: The square root of variance. It measures the average deviation of returns from the mean in the same units as the returns.
- **Formula**: $\sigma = \sqrt{\sigma^2}$

c. Range

- **Definition**: The difference between the highest and lowest returns.
- **Formula**:

Range=Maximum Return−Minimum ReturnRange=Maximum Return−Minimum Return

- **Example**: For returns {5%,10%,−2%,7%}{5%,10%,−2%,7%}:

  Range=10%−(−2%)=12%Range=10%−(−2%)=12%

The range can be considered as a poor measure of dispersion because it becomes larger as the sample size increases. The variance is the most frequently used parameter in the measure of dispersion.

1.2. Measuring Portfolio Risk and Return

The risk and return of a portfolio are quantified using two key metrics: the expected rate of return and the variance (or standard deviation) of the combination of financial assets within the portfolio. The expected return represents the average outcome investors anticipate, while the variance measures the dispersion of returns around this average, reflecting the portfolio's risk level.

In the mean-variance framework, developed by Harry Markowitz in his seminal work on Modern Portfolio Theory (MPT), only the mean (expected return) and variance (risk) are required to describe the end-of-period wealth distribution. This approach assumes that investors base their decisions solely on these two parameters, aiming to maximize returns for a given level of risk or minimize risk for a given level of return.

However, this framework relies on specific assumptions. Unless investors have quadratic utility functions (which are rarely the case in practice) or are exposed to minimal risk, it becomes necessary to assume that portfolio returns follow a normal distribution. The normal distribution is characterized by its symmetry and the fact that it is fully described by its mean (μ) and variance (σ^2). The probability density function (PDF) of a normal distribution is given by:

$$f(\hat{R}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\hat{R}-\mu}{\sigma}\right)^2} \; Where \; E(\hat{R}) = \mu \text{ and } V(\hat{R}) = \sigma^2$$

This assumption simplifies the analysis, as it allows investors to focus solely on the mean and variance without considering higher moments of the distribution, such as skewness (asymmetry) or kurtosis (fat tails). However, in reality, financial returns often deviate from normality, exhibiting features like fat tails and skewness, which can lead to underestimation of extreme risks (Mandelbrot, 1963), (Taleb, 2007).

2. Two-asset and multi-asset portfolio selection

2.1. Calculating the Mean and Variance of a Two-Asset Portfolio

When constructing a portfolio with two financial assets, the rate of return for each asset is denoted as **X** for the first asset and **Y** for the second. Let **a** represent the proportion of the initial wealth $\mathbf{W_0}$ invested in the first asset. Consequently, the rate of return of the portfolio $\mathbf{R_P}$, its expected return $\mathbf{E(R_P)}$, and its variance $\mathbf{V(R_P)}$ can be calculated as follows:

2.1.1. Portfolio Rate of Return: This equation shows that the portfolio return is a weighted average of the returns of the two assets. $Rp = aX + (1-a)Y$

2.1.2. **Expected Return of the Portfolio**: Here, μ_X and μ_Y are the expected returns of assets **X** and **Y**, respectively:

$E(Rp) = aE(X) + (1-a)E(Y)$ **or** $\mu p = a\mu_X + (1-a)\mu$

2.1.3. **Variance of the Portfolio**:

$V(Rp) = a^2 V(X) + (1-a)^2 V(Y) + 2a(1-a)\mathrm{Cov}(X,Y)$ Or, in terms of standard deviations (**σ**):

$$\sigma^2 p = a^2 \sigma^2_X + (1-a)^2 \sigma^2_Y + 2a(1-a)\mathrm{Cov}(X,Y).$$

The variance of the portfolio depends not only on the variances of the individual assets but also on their covariance, which measures how the returns of the two assets move together. The proportion **a** is typically a positive value between 0 and 1. However, if **short selling** is allowed, **a** can take values outside this interval. Short selling involves selling an asset (e.g., asset **X**) that the investor does not own, using the proceeds to invest in another asset (e.g., asset **Y**). At the end of the investment period, the investor must repurchase the short-sold asset to return it to the lender. This strategy can amplify returns but also increases risk.

2.1.4.The correlation coefficient:

Using the correlation coefficient: $\rho_{XY} = Cov(X,Y)/\sigma_X\sigma_Y$,the variance of the portfolio rate can be rewritten:

$$\sigma^2(\hat{R}_p) = a^2\,\sigma^2(X) + (1-a)^2\,\sigma^2(Y) + 2\,a\,(1-a)\,\rho\,\sigma(X)\,\sigma(Y)$$

When two assets are perfectly correlated, their correlation coefficient ρ is either +1 (perfect positive correlation) or −1 (perfect negative correlation). Let's analyze the implications for portfolio construction in both cases.

a. Perfect Positive Correlation (ρ=+1):

When two assets are perfectly positively correlated, their returns move in the same direction proportionally. The portfolio variance and risk reduction are limited because there is no diversification benefit.

from $\sigma^2(\hat{R}_p) = a^2\,\sigma^2(X) + (1-a)^2\,\sigma^2(Y) + 2\,a\,(1-a)\,\rho\,\sigma(X)\,\sigma(Y)$

We get : $\sigma^2(\hat{R}_p) = (a\sigma_X + (1-a)\sigma_Y)^2$

And consequently :

$$a^* = \frac{\sigma^2(Y) - \rho\sigma(X)\sigma(Y)}{\sigma^2(X) + \sigma^2(Y) - 2\rho\sigma(X)\sigma(Y)} = \frac{\sigma(Y)(\sigma(Y) - \sigma(X))}{(\sigma(Y) - \sigma(X))^2} = \frac{\sigma(Y)}{(\sigma(Y) - \sigma(X))}$$

with: $\sigma_X \neq \sigma_Y$

We can also use the two equations to determine the expression of μp as a function of σp

$$\mu_p = a\,\mu_X + (1-a)\,\mu_Y = a(\mu_X - \mu_Y) + \mu_Y \quad (1)$$
$$\sigma_p = (a\sigma_X + (1-a)\sigma_Y) = a(\sigma_X - \sigma_Y) + \sigma_Y \quad (2)$$

Therefore

$$\frac{\mu_p - \mu_Y}{(\mu_X - \mu_Y)} = \frac{\sigma_p - \sigma_Y}{(\sigma_X - \sigma_Y)} = a \Rightarrow \mu_p = \frac{\sigma_p\,(\mu_X - \mu_Y)}{(\sigma_X - \sigma_Y)} - \frac{\sigma_Y\,(\mu_X - \mu_Y)}{(\sigma_X - \sigma_Y)} + \mu_Y$$

and we can see that μp is a linear function of σp and its first derivative with respect to σp is a constant that is equal to $(\mu_X - \mu_Y)/(\sigma_X - \sigma_Y)$

b. **Perfect negative Correlation (ρ=-1): occurs when two variables move in exactly opposite directions. Because of the absolute value of the standard deviation, we need to consider two situations:**

A. $\underline{a\sigma_X - (1-a)\sigma_Y \geq 0 \Rightarrow \sigma_p = (a\sigma_X - (1-a)\sigma_Y) \text{ and } a \geq \sigma_Y / \sigma_X + \sigma_Y}$

Using the same steps as (5-1), we find:

$$\mu_p = \frac{\sigma_p\ (\mu_X - \mu_Y)}{(\sigma_X + \sigma_Y)} + \frac{\sigma_Y\ (\mu_X - \mu_Y)}{(\sigma_X - \sigma_Y)} + \mu_Y$$

We can see that μ_p is a linear function of σp and its first derivative with respect to σp is a constant that is equal to $(\mu_X - \mu_Y)/(\sigma_X - \sigma_Y)$

B. $\underline{a\sigma_X - (1-a)\sigma_Y \leq 0 \Rightarrow \sigma_p = -(a\sigma_X - (1-a)\sigma_Y) \text{ and } a \leq \sigma_Y / \sigma_X + \sigma_Y}$

Using the same steps, we find:

$$\mu_p = -\frac{\sigma_p\ (\mu_X - \mu_Y)}{(\sigma_X + \sigma_Y)} - \frac{\sigma_Y\ (\mu_X - \mu_Y)}{(\sigma_X - \sigma_Y)} + \mu_Y$$

We can see that μp is a linear function of σp and its first derivative with respect to σp is a constant that is equal to $-(\mu_X - \mu_Y)/(\sigma_X + \sigma_Y)$

❖ **Interpretation:**

- The opportunity set is represented by two line segments in the (σ_p,μ_p) plane. These segments correspond to the two cases:

1. $a\sigma_X - (1-a)\sigma_Y \geq 0$
2. $a\sigma_X - (1-a)\sigma_Y \leq 0$

- The slopes of these segments are:

$$\frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y} \quad \text{and} \quad -\frac{\mu_X - \mu_Y}{\sigma_X + \sigma_Y}$$

These slopes are constants, indicating that μp is a linear function of σp.

- The three key points in the (σ_p,μ_p) plane are:

1. Asset X: (σ_X, μ_X)
2. Asset Y: (σ_Y, μ_Y)
3. The minimum variance portfolio (not explicitly derived here but implied).

c. **Two independent assets (ρ=0)**

$$\mu_p = a\,\mu_X + (1\text{-}a)\,\mu_Y = a(\mu_X - \mu_Y) + \mu_Y \quad \text{and} \quad \sigma_p^2 = a^2\sigma_X^2 + (1-a)^2\sigma_Y^2$$

Therefore from the equation portfolio expected return above, we have:

$$\frac{\mu_p - \mu_Y}{(\mu_X - \mu_Y)} = a \Rightarrow \sigma_p^2 = \frac{\sigma_X^2\,(\mu_p - \mu_Y)^2}{(\mu_X - \mu_Y)^2} + \frac{\sigma_Y^2\,(\mu_p - \mu_X)^2}{(\mu_X - \mu_Y)^2}$$

If we develop some more, it is easy to see that this is a parabola in the (μ, σ2) plan:

$$\sigma_p^2 = \mu_p^2\,\frac{(\sigma_X^2 + \sigma_Y^2)}{(\mu_X - \mu_Y)^2} - 2\mu_p\,\frac{(\mu_Y\sigma_X^2 - \mu_X\sigma_Y^2)}{(\mu_X - \mu_Y)^2} + \frac{(\mu_Y^2\sigma_X^2 - \mu_Y^2\sigma_Y^2)}{(\mu_X - \mu_Y)^2}$$

The relationship between the expected return μ_p and the variance $\sigma_p{}^2$ of the portfolio can be represented as a parabola in the (μ, σ^2) plane. The curve of this parabola goes through points corresponding to the individual assets X and Y, as well as the minimum variance portfolio C.

d. **X and Y assets such that: (i.e.,ρ≠0,1,−1)**

From the expected return equation, we can solve for a: $a=\mu X-\mu Y/\mu p-\mu Y$.
Substitute a into the Risk Equation:

$$\sigma_p^2 = \frac{\sigma_X^2\,(\mu_p - \mu_Y)^2}{(\mu_X - \mu_Y)^2} + \sigma_Y^2\Big(1 - \frac{\mu_p - \mu_Y}{\mu_X - \mu_Y}\Big)^2 + 2\,\frac{\mu_p - \mu_Y}{\mu_X - \mu_Y}\Big(1 - \frac{\mu_p - \mu_Y}{\mu_X - \mu_Y}\Big)\rho_{XY}\sigma_X\sigma_Y$$

We see again that this curve represents a parabola in the (μ, σ^2) plan.

The portfolio opportunity set (Λ) represents all possible combinations of two assets, X and Y, in terms of their expected return (μ) and risk (σ). When plotted in the standard deviation-mean (σ, μ) plane, this set forms a line segment connecting the two assets, A (Asset X) and B (Asset Y). The position of any portfolio along this line segment is determined by the weight a assigned to Asset X, where $0 \le a \le 1$. The weight assigned to Asset Y is $1-a$.

The set of all possible portfolios formed by varying a from 0 to 1 is represented as:

$\Lambda=\{(a,1-a)|\mu p=f(\sigma p)\}$

This set traces a curve (or line segment) in the (σ, μ) plane, connecting the points corresponding to Asset X (a=1) and Asset Y (a=0).

The graph below illustrates the portfolio opportunity set in the risk-return space (σ,μ), highlighting how different correlation coefficients between two assets (X and Y) affect the shape of the set of possible portfolios.

- The horizontal axis (σ) represents risk (measured by standard deviation).
- The vertical axis (μ) represents expected return**.**
- The two assets, X and Y, are marked with crosses, and the curves show the different combinations of portfolios formed by varying the weights of X and Y.

The red lines correspond to different correlation values:

- **ρ = +1 (perfect positive correlation)**: The opportunity set is a straight line between X and Y, offering no diversification benefit.
- **ρ = 0 (zero correlation)**: The opportunity set becomes a curve, showing moderate risk reduction through diversification.
- **ρ = −1 (perfect negative correlation)**: The curve becomes a V-shape, allowing for maximum diversification and even complete risk elimination at the optimal weight combination.

The graph also includes indifference curves, which represent investor preferences: higher curves indicate combinations of risk and return that provide greater utility.

This visual is essential in Modern Portfolio Theory, as it demonstrates how correlation is a key driver of portfolio risk, and how investors choose optimal portfolios based on their risk-return preferences.

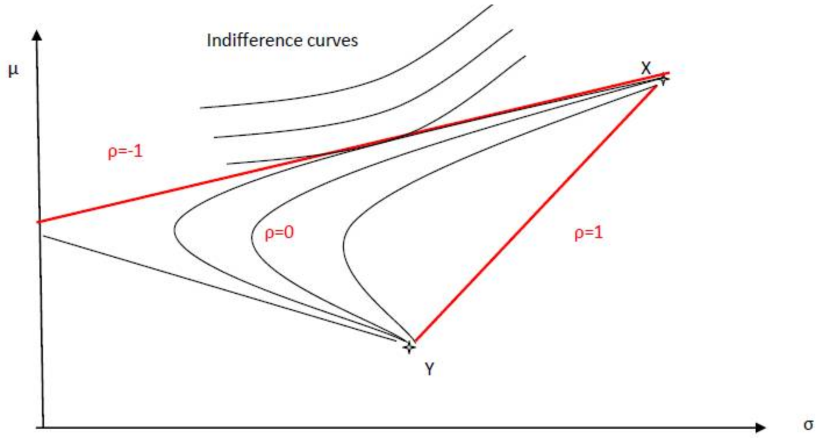


FIG (1.1) :Portfolio Opportunity Set and the Effect of Correlation on Risk and Return

Problem 1: Consider a risk-neutral individual faced with two perfectly correlated financial assets, **A** and **B**, where:

- $E(B)=k1\cdot E(A)$
- $V(B)=k2\cdot V(A)$

Determine the opportunity set and the optimal solution for the following values of

$(k1,k2)$:{(1,1),(1,2) ,(2,1), (2,2)}

Consider two types of investors: Risk-neutral investors &Risk-averse investors. Use the (σ,μ) plane to represent the assets and portfolios.

Problem 2:

Assume two assets, **C** and **D**, with the following characteristics:

- **Asset C**: Expected return $E(C)$=15%, Standard deviation $\sigma(C)=12$
- **Asset D**: Expected return $E(D)$=5%, Standard deviation $\sigma(D)=6$
- **Covariance between C and D**: $\text{Cov}(C,D)=-20$

Let **a** represent the proportion of the initial wealth $W0$ invested in Asset C. Answer the following questions:

1. **Calculate the expected return $E(Rp)$ and variance $V(Rp)$ of the portfolio for a=0.6.**

2. **What is the distribution of the portfolio rate of return *Rp*, and what are its parameters?**
3. **Assume $a=-0.4$ and $W0=500$. Compute $E(Rp)$, $V(Rp)$, $E(W)$, and $V(W)$. Interpret the results.**

2.2. Optimal portfolio choice: many assets

2.2.1 The portfolio return *Rp*

Let $Ri \sim N(\mu i, \sigma i2)$ represent the rate of return of asset *i*, where:

- μi is the expected return of asset *i*,
- $\sigma^2 i$ is the variance of the return of asset *i*,
- ρij is the correlation between the returns of assets *i* and *j*, defined as:

  $\rho ij = \text{cov}(Ri,Rj)/\ \sigma i \sigma j$

Let x_i represent the weight of asset *i* in the portfolio, such that: $\sum_{i=1}^{N} xi = 1$

The **portfolio return** *Rp* is given by:

a- Scalar Notation: $Rp=x1R1+x2R2+\cdots+xNRN= \sum_{i=1}^{N} xi\ Ri = \sum_{i=1}^{N} xi\ Ui$

b- Matrix Notation In matrix notation:

- Let $\mathbf{X}'=(x1,x2,\ldots,xN)$ be the column vector of portfolio weights.
- Let $\mathbf{R}'=(R1,R2,\ldots,RN)$ be the column vector of asset returns.
- Let $\mu=(\mu 1,\mu 2,\ldots,\mu N)$ be the column vector of expected returns.

The portfolio return *Rp* can be written as: $Rp=\mathbf{X}'\mathbf{R}$

The expected portfolio return $E(Rp)$ in matrix notation is:

$E(Rp)=E(\mathbf{X}'\mathbf{R})=\mathbf{X}'E(\mathbf{R})=\mathbf{X}'\mu$

Both notations express the same concept: the expected return of the portfolio is the weighted sum of the expected returns of the individual assets. Matrix notation is more compact and convenient, especially when dealing with many assets.

2.2.2. Portfolio Variance

a. Portfolio Variance in Scalar Notation: For a portfolio with N assets, The portfolio variance $V(Rp)$ in scalar notation is:

$V(Rp)=V(x1R1+x2R2+\cdots+xNRN)$

Expanding this, we get: $V(Rp)=\sum_{i=1}^{N} xi^2\sigma^2 i+\sum\sum xixj\sigma ij$

This can also be written as: $V(Rp)=\sum\sum xixj\sigma ij$ where:

- $\sigma^2 ij=\sigma^2 i$ when $i=j$ (variance terms),
- $\sigma ij=\mathrm{cov}(Ri,Rj)$ when $i\neq j$ (covariance terms).

b- Portfolio Variance in Matrix Notation

Let Σ be the variance-covariance matrix of the asset returns, where:

$$\boldsymbol{\Sigma}=\begin{bmatrix}\sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1N}\\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_N^2\end{bmatrix}$$

$V(Rp)=V(X'R)=X'\Sigma X$

2.3. The opportunity set with N risky assets

The opportunity set with N risky assets refers to the set of all possible portfolios that can be formed by combining these N assets in different proportions. This set is often visualized in terms of risk and return, and it provides the foundation for modern portfolio theory (MPT), developed by Harry Markowitz.

When considering portfolios with many assets, we can discover the opportunity set and efficient set if we know the expected returns and the variances of individual assets as well as the covariances between each pair of assets. There were not many assets to consider in the hedging example, but an investor can choose literally any combination of securities. This requires a great deal of information. The New York Stock Exchange alone lists at least 2000 securities. To determine the opportunity set it would be necessary to estimate 2000 mean returns, 2000 variances, and 1,999,000 covariances." Fortunately, we shall soon see that there are ways around this computational nightmare (fig 2)

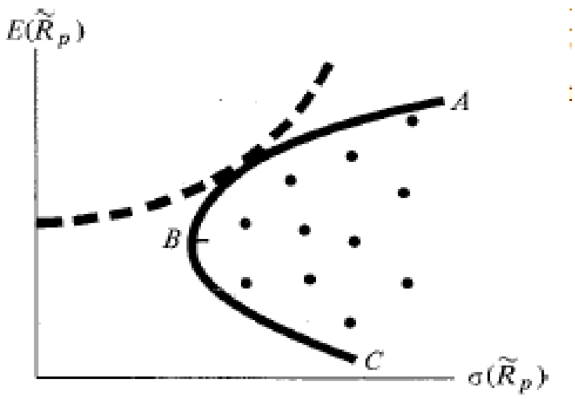


Fig (1.2): The investment opportunity set with many risky assets

3. The minimum variance portfolio

The Minimum Variance Portfolio (MVP) is the portfolio allocation between a set of assets that minimizes overall risk (portfolio variance) without considering expected returns. It is a foundational concept in Modern Portfolio Theory **(MPT)**.

The portfolio variance is given by: $\boldsymbol{\sigma p}^2 \quad = \sum_{ij=0}^{n} \boldsymbol{wiwj\sigma ij}$

where:

- wi = weight of asset i,
- σij = covariance between assets i and j ($\sigma ii = \sigma i^2$).

Optimization Problem:

Minimize σp^2 subject to: $\sum_{i=1}^{N} wi$ =1(fully invested)

(if $wi \geq 0$ short-selling is prohibited.)

3.1. Minimum Variance Portfolio for Two Assets

Given two assets:

- **Asset A**: Expected return = μA, Standard deviation = σA
- **Asset B**: Expected return = μB, Standard deviation = σB
- **Covariance** between A and B = σAB
- **Correlation** = $\rho AB = \frac{\sigma AB}{\sigma A \sigma B}$

Let w = weight of Asset A in the portfolio. Then, the portfolio variance is:

$\sigma p^2 = w^2 \sigma A^2 + (1-w)^2 \sigma B^2 + 2w(1-w)\sigma AB$

Find the weight w^* that **minimizes** σp^2.

Take the derivative of σp^2 with respect to w and set it to zero:

$\frac{d\sigma 2p}{dw} = 2w\sigma^2 A - 2(1-w)\sigma^2 B + 2(1-2w)\sigma AB = 0$

Let's expand and rearrange the equation:

$2w\sigma^2 A - 2\sigma^2 B + 2w\sigma^2 B + 2\sigma AB - 4w\sigma AB = 0$

Combine like terms:

$2w\sigma^2 A + 2w\sigma^2 B - 4w\sigma AB = 2\sigma^2 B - 2\sigma AB$

Factor out $2w$ on the left:

$2w(\sigma^2 A + \sigma^2 B - 2\sigma AB) = 2(\sigma^2 B - \sigma AB)$

Divide both sides by 2:

$w(\sigma^2 A + \sigma^2 B - 2\sigma AB) = \sigma^2 B - \sigma AB$

Finally, solve for w:
$$w^* = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$$

$w^* = (\sigma^2 B - \sigma AB) / \sigma^2 A + \sigma^2 B - 2\sigma AB$

Recall that $\sigma AB = \rho AB \sigma A \sigma B$. Substituting:
$$w^* = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}$$

$w*=\sigma^2A+\sigma^2B-2\rho AB\sigma A\sigma B\sigma B2-\rho AB\sigma A\sigma B$

The optimal weight for asset A in the two-asset portfolio is:

$$w^* = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B}$$

This weight minimizes the portfolio variance. The weight for asset B is simply $1-w*$.

3.2. Multi-Asset Portfolio selection:

We seek to find the optimal portfolio weights w that minimize portfolio variance for a given target expected return μ*, subject to full investment (weights sum to 1).

Formally: $\min w^T\Sigma w$

Subject to:

1. Return constraint: $w^T\mu=\mu*$
2. Budget constraint: $w^T1=1$

Where:

- $w=[w1,w2,...,wn]T$ = vector of portfolio weights
- Σ = $n\times n$ covariance matrix of asset returns
- $\mu=[\mu 1,\mu 2,...,\mu n]^T$ = vector of expected returns
- $1=[1,1,...,1]^T$ = unit vector
- $\mu*$ = target expected return

To solve this constrained optimization problem, we use the method of Lagrange multipliers, introducing two Lagrange multipliers ($\lambda 1$ and $\lambda 2$) for the two constraints.

$$L(w,\lambda_1,\lambda_2)=w^T\Sigma w+\lambda_1(w^T\mu-\mu*)+\lambda 2(w^T1-1)$$

Let us define the following intermediate scalars:

- $A=1^T\Sigma^{-1}1$
- $B=1^T\Sigma^{-1}\mu$

- $C = \mu^T \Sigma^{-1} \mu$
- $D = AC - B^2$

Then, the solution for the optimal weights w* is:

$$w^* = \Sigma^{-1} \quad \frac{C - \mu^* B}{D}\mathbf{1} + \frac{\mu^* A - B}{D}\mu$$

This gives the vector of optimal weights that minimize portfolio variance for a given target expected return μ*.

4. Frontier portfolios set (Λ_f)

The minimum variance opportunity set is the collection of all portfolios that minimize the portfolio variance $\sigma^2 p$ for a given level of expected return μp, with Λ_f is a subset of Λ. Mathematically, it is the set of portfolios that solve the optimization problem: min $\sigma^2 p$ subject to μ_p= k

The following figure shows the minimum-variance frontier and identifies the efficient set of portfolios that offer the highest return for a given level of risk.

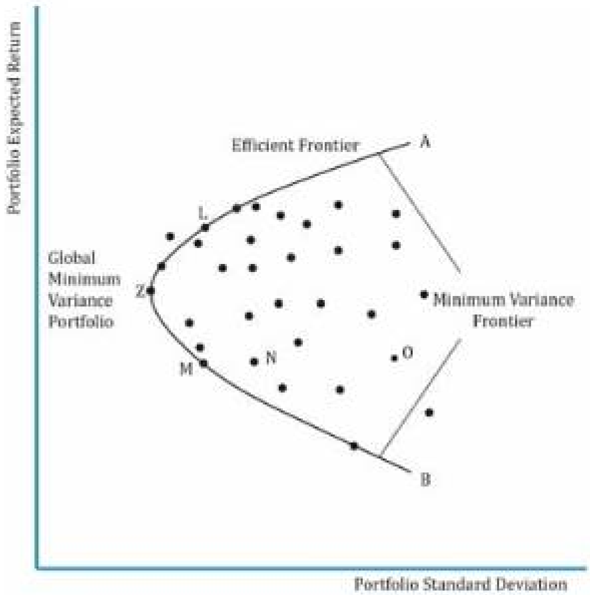


Fig (1.3): The Minimum-Variance Frontier and the Efficient Frontier

5. Efficient Set with Two Risky Assets (No Risk-Free Asset)

The Efficient Set (or Efficient Frontier) is a collection of optimal portfolios that offer the highest expected return for a given level of risk (standard deviation), or the lowest risk for a given expected return.

When constructing the efficient set with two risky assets and no risk-free asset, the goal is to identify the set of portfolios that offer the highest expected return for a given level of risk (variance) or the lowest risk for a given level of expected return. This set is known as the efficient frontier**.**

This set is obtained by solving the following mathematical programming problem: max μp subject to $\sigma^2 p = \text{k}$.

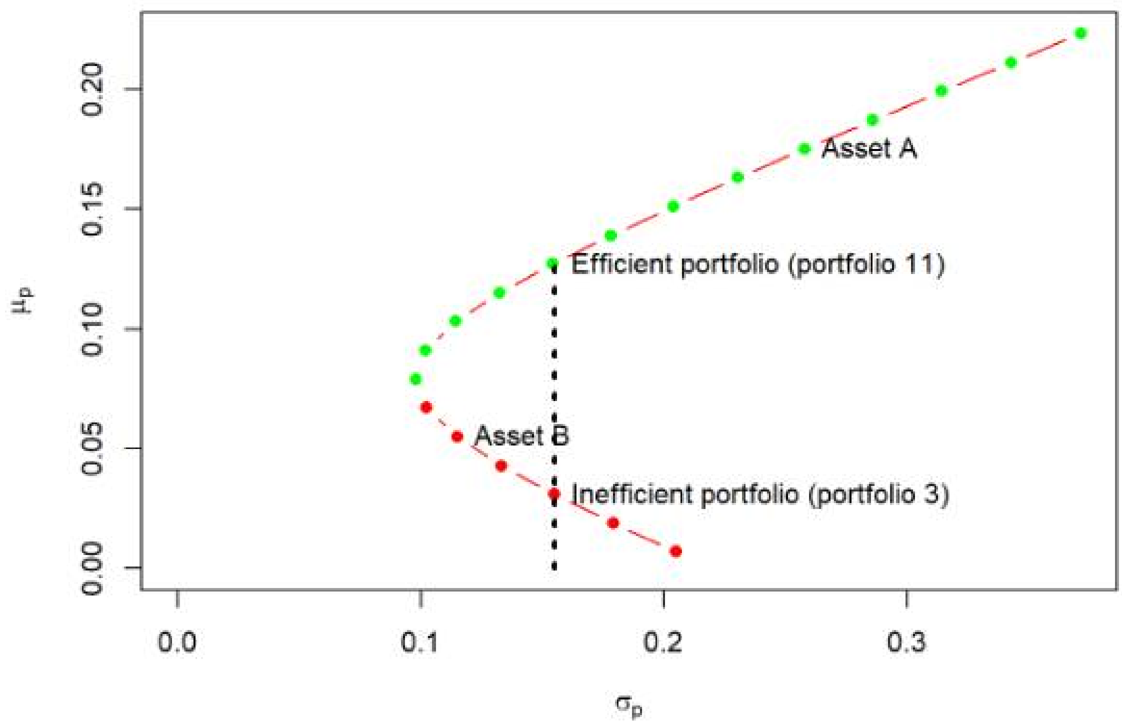


Fig (1.4): the efficient and inefficient portfolios

Figure (4) shows the efficient and inefficient portfolios. The efficient portfolios are those that have the highest expected return for a given standard deviation value. These portfolios are the green dots starting with the global minimum variance portfolio at the tip of the Markowitz bullet. The inefficient portfolios are the red dots below the global minimum variance portfolio. For example, Figure (4) shows two feasible portfolios, portfolio 3 and portfolio 11, with the same standard deviation value but different expected return values. Portfolio 11 has the highest expected return and so is the efficient portfolio. Portfolio 3, with the lower expected return, is the inefficient portfolio.

Because there is no risk free asset, the investor does not have the possibility to lend or to borrow. Therefore, this individual will choose his optimal portfolio of risky assets in a world where there is no possibility of exchange. Note that, in this case, the objects of choice are risk and return. The investor's optimal portfolio is that where his subjective marginal rate of substitution between risk and return is equal to the objectively determined marginal rate of transformation, along his mean variance or mean standard deviation opportunity set, between risk and return:

$$\mathrm{MRS}_{\mu_p}^{\sigma_p} = \mathrm{MRT}_{\mu_p}^{\sigma_p}$$

6. Risk-return trade-off in efficient portfolios

The risk-return trade-off in efficient portfolios is the fundamental principle that in the context of optimally constructed portfolios, higher expected returns can *only* be achieved by accepting a higher level of risk. This is not the same as the trade-off for individual assets, where a high-risk asset might yield low returns. The "efficient" qualifier is critical, as it refers to portfolios that lie on the Efficient Frontier—the set of portfolios that offer the highest possible expected return for a given level of risk, or conversely, the lowest possible risk for a given level of expected return.

Let a be the weight of the **risky asset (X)** in the portfolio, and 1−a be the weight of the **risk-free asset (Rf)**.

- **Expected Return of the Portfolio (μp):** $\mu p = a \cdot \mu_X + (1-a) \cdot R_f$ (1)
- **Risk of the Portfolio (σp):** Since the risk-free asset has no risk ($\sigma Rf=0$), the portfolio risk depends only on the risky asset: $\sigma p = a \cdot \sigma_X$ (2)

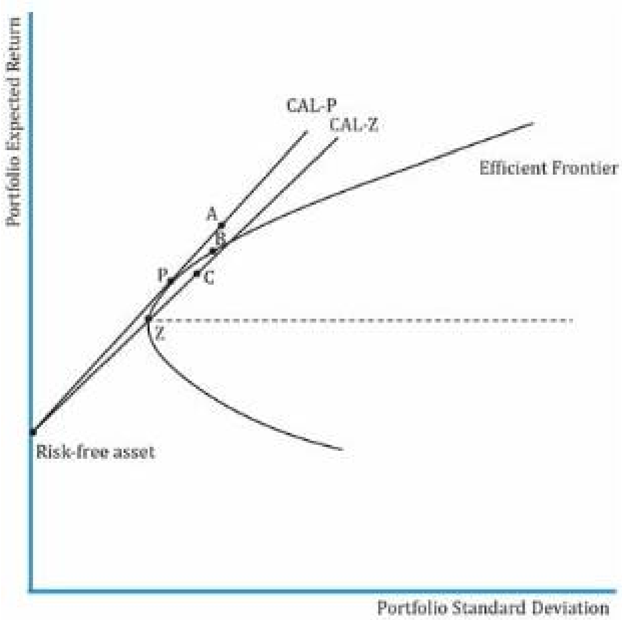
from (1) & (2):

$$\mathrm{a} = \frac{\mu_\mathrm{p} - R_\mathrm{f}}{(\mu_\mathrm{X} - R_\mathrm{f})} = \frac{\sigma_\mathrm{p}}{\sigma_\mathrm{X}}$$

The **efficient set** in this case is a straight line called the **Capital Allocation Line (CAL)**, which represents all possible combinations of the risky asset and the risk-free asset. The CAL is defined by the equation:

$$\mu p = Rf + \frac{(\mu X - Rf)}{\sigma X} \sigma p$$

The graph (5) expands upon the basic Mean-Variance Efficient Frontier by introducing a risk-free asset (such as a short-term government bond). The ability to lend and borrow at this risk-free rate fundamentally changes the investment landscape and allows us to identify a single optimal risky portfolio.



graph (1.5): The Capital Allocation Line (CAL) and the Optimal Portfolio

Graph interpretation:

- The curved line represents all efficient portfolios composed *solely of risky assets.* While optimal for their given risk level, they are suboptimal once a risk-free asset is available for diversification.
- The straight lines (CAL-Z and CAL-P) depict the new, superior risk-return opportunities created by combining the risk-free asset with a specific risky portfolio (Z or P). The slope of each CAL represents its Sharpe Ratio (reward-to-variability ratio).
- Portfolio P is the tangency portfolio where the capital allocation line just touches the efficient frontier. This line, CAL-P, has the steepest possible slope, meaning it offers

the highest expected return per unit of risk. Consequently, CAL-P becomes the optimal Capital Allocation Line**.**

- A fundamental implication is the Separation Theorem: the identity of the optimal risky portfolio (P) is determined solely by capital market expectations (the efficient frontier and the risk-free rate) and is *separate* from an investor's individual risk aversion.
- An investor's risk tolerance then dictates their *position along* CAL-P:
  - ✓ **Lending (Conservative):** Points between the risk-free asset and P represent a mix of the risk-free asset and portfolio P.
  - ✓ **Full Investment:** Point P represents a 100% allocation to the optimal risky portfolio.
  - ✓ **Borrowing (Aggressive):** Points on the extension beyond P represent using leverage (borrowing at the risk-free rate) to purchase more of portfolio P.

5.1. Slope of the CAL: $\frac{(\mu X - Rf)}{\sigma X}$, also known as the Sharpe ratio. It measures the excess return per unit of risk.

This slope is known as the **Sharpe ratio**, which measures the excess return per unit of risk. The Sharpe ratio is a key metric for evaluating the performance of an investment or portfolio. A higher Sharpe ratio indicates a more favorable risk-adjusted return.

5.2. Implications of the CAL

a. **Risk-Return Tradeoff**: The CAL illustrates the tradeoff between risk and return. As you increase the weight of the risky asset (a) in the portfolio, both the expected return (μp) and the risk (σp) increase.

b. **Optimal Portfolio**: Investors can choose any point along the CAL based on their risk tolerance. More risk-averse investors will choose a portfolio with a higher weight in the risk-free asset, while more risk-tolerant investors will choose a portfolio with a higher weight in the risky asset.

c. **Leverage**: If an investor is willing to take on more risk, they can borrow at the risk-free rate and invest more than 100% of their capital in the risky asset. This is represented by points on the CAL beyond the risky asset (X).

Example: Let's assume: $E(Rr)$=10%, σr=20, Rf=3%

- Calculate Expected Return and Risk for Different Weights (0%, 50%,100%) and draw the efficient set.

Solution

1. **Weight in Risky Asset (a)= 0%**:
   - $E(Rp)=0 \cdot 10\%+1 \cdot 3\%=3\%$
   - $\sigma p=0 \cdot 20\%=0\%$
2. **Weight in Risky Asset (a) = 50%**:
   - $E(Rp)=0.5 \cdot 10\%+0.5 \cdot 3\%=6.5\%$
   - $\sigma p=0.5 \cdot 20\%=10\%$
3. **Weight in Risky Asset (a) = 100%**:
   - $E(Rp)=1 \cdot 10\%+0 \cdot 3\%=10\%$
   - $\sigma p=1 \cdot 20\%=20\%$

5.3 Extending Beyond the Risky Asset (Leverage)

The segment of the Capital Market Line (CML) between the risk-free asset (Rf) and the market portfolio (M) represents **lending portfolios**. In this region, an investor allocates their own wealth between these two assets without using leverage. By holding a portion in the risk-free asset, the investor is effectively "lending" capital at the risk-free rate, resulting in a portfolio with lower risk and return than the market portfolio (M).

The extension of the CML beyond point M represents **borrowing portfolios**. This assumes the investor can borrow additional funds at the risk-free rate to invest more than 100% of their own wealth in the optimal risky market portfolio (M). This use of financial leverage creates portfolios with higher expected returns, but also higher risk, than the market portfolio itself.

The graph () illustrates the full scope of the Capital Market Line (CML) under the key assumption of a single, universal risk-free rate for both lending and borrowing. The CML depicts the set of optimal portfolios available to all investors, demonstrating how the combination of the risk-free asset and the market portfolio (M) creates a linear, efficient risk-return trade-off.

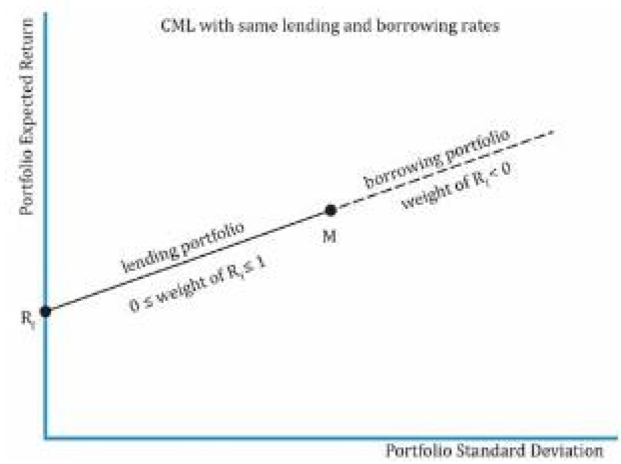


Fig (1.6): CML-Leveraged portfolios

- The line originates at the risk-free rate (*Rf*), where the portfolio is 100% invested in the risk-free asset. As we move along the CML towards point M, the investor's wealth is increasingly allocated to the risky market portfolio, with the segment between *Rf* and M representing **lending portfolios** (positive allocation to the risk-free asset).

- The critical insight is extended beyond point M, where the line continues upward. This segment represents **borrowing, or leveraged, portfolios**. Here, an investor borrows money at the risk-free rate to invest more than 100% of their own capital in the market portfolio (M), resulting in a negative weight in the risk-free asset. This leverage increases both the expected return and the risk of the portfolio. The slope of the entire CML, given by $E(Rm)-Rf/\ \sigma m$, represents the market price of risk, indicating the additional return offered by the market for each additional unit of risk undertaken.

Example 1:

- **Weight in Risky Asset (a) = 150%** (Borrowing 50% at *Rf*):
  - $E(Rp)=1.5\cdot 10\%+(-0.5)\cdot 3\%=15\%-1.5\%=13.5\%$
  - $\sigma p=1.5\cdot 20\%=30\%$

If short sales are allowed, then $\sigma(\hat{R}_p) = |a|\,\sigma(X)$ and therefore there are two situations:

First situation: "a" is positive, then $$\mu_p = \frac{\sigma_p\,(\mu_X - R_f)}{\sigma_X} + R_f$$

Second situation: "a" is negative, then $\mu_p = -\frac{\sigma_p(\mu_X - R_f)}{\sigma_X} + R_f$

Therefore, when short sales are allowed, the opportunity set is represented by two line segments.

Example 2:

Weight in Risky Asset (w) = -50% (Short Selling):

- o E(Rp)=−0.5·10%+1.5·3%=−5%+4.5%=−0.5%
- o σp=0.5·20%=10%

Interpreting the Opportunity Set

- Upper Segment (Leveraging): Higher expected returns with higher risk.
- Lower Segment (Short Selling): Lower or negative expected returns with increasing risk.

Investors can choose any point along these segments based on their risk tolerance and return objectives.

Exercises:

Exercise 1

You are given:

Asset	E(R)	σ	ρ_{12}	ρ_{13}	ρ_{23}
A	12%	20%	—	—	—
B	8%	10%	0.3	—	—
C	15%	25%	0.1	0.5	0.2

Tasks:

1. Compute the variance–covariance matrix.
2. Find the minimum variance portfolio.
3. Determine the efficient frontier for combinations of A, B, and C.
4. Plot (μ, σ) using Excel or Python.
5. Compute the Sharpe ratio for each point.

Exercise 2

Suppose an investor has utility function: $$U = E(R_p) - \frac{A}{2}\sigma_p^2$$

With risk aversion coefficient A=3.

- Find the optimal portfolio given:
  - o Risk-free rate Rf=2%
  - o Risky asset E(R)=10%, σ =15%.
- Plot indifference curves and the Capital Allocation Line.
- Find the optimal weight in the risky asset a* and expected utility.

Exercise 3

Two assets: E(R1)=14%, σ1=20%, E(R2)=8%, σ2=10%, Correlation ρ=−0.5

1. Compute the weight w* that minimizes portfolio variance.
2. Compute σp and E(Rp).
3. Compare results for ρ=+0.5,0,−1. Discuss the diversification effect.

Exercise 4

Given three risky assets with:

$$\mu = \begin{bmatrix} 0.12 \\ 0.10 \\ 0.07 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.04 & 0.02 & 0.01 \\ 0.02 & 0.03 & 0.015 \\ 0.01 & 0.015 & 0.02 \end{bmatrix}$$

1. Compute Σ^{-1},A, B, C, and D.
2. Find the global minimum variance portfolio (no target return).
3. Then compute the weights for a target return of 10%.

Chapter 2: Advanced Portfolio Models & Market Equilibrium

The inclusion of a risk-free asset in portfolio optimization reshapes the efficient frontier, replacing the Markowitz bullet with the linear Capital Market Line (CML)a cornerstone of modern financial theory. This framework, pioneered by (Tobin, 1958), introduces the Separation Theorem, which decouples portfolio selection into two distinct phases: (1) identifying the universal tangency portfolio of risky assets, and (2) tailoring risk exposure by combining this portfolio with risk-free borrowing or lending. These insights not only provide the theoretical foundation for the Capital Asset Pricing Model (CAPM) but also inform practical strategies in institutional asset management. This section examines their mathematical derivations, equilibrium implications, and critical role in bridging theoretical finance with real-world investment decisions.

1. Introduction of the risk-free asset

1.1 The opportunity set with N risky assets and one risk free asset

When you introduce a **risk-free asset** into the opportunity set with *N* risky assets, the investment possibilities expand significantly. This combination forms the basis of the Capital Market Theory and leads to the concept of the Capital Market Line (CML)**.** The risk-free asset allows investors to borrow or lend at the risk-free rate, which changes the shape of the efficient frontier and introduces new optimal portfolios.

1.2 The Efficient set with N Risky Assets and one Risk-Free Asset:

When constructing the efficient set with one risky asset and one risk-free asset, the goal is to identify the set of portfolios that offer the highest expected return for a given level of risk (variance) or the lowest risk for a given level of expected return. This set is known as the Capital Allocation Line (CAL)**.**

Once the risk-free asset is introduced into the analysis, the problem of portfolio selection is simplified. If, as before, we assume that the borrowing rate equals the lending rate, we can draw a straight line between any risky asset and the risk-free asset. Points along the line represent portfolios consisting of combinations of the risk-free and risky assets. Several possibilities are graphed in Fig. (5-1). Portfolios along any of the lines are possible, but only one line dominates. All investors will prefer combinations of the risk-free asset and portfolio M on the efficient set. (Why?)

These combinations lie along the positively sloped portion of line NMRf0. Therefore the efficient set (which is represented by line segment RfMN) is linear in the presence of a risk-free asset. All an investor needs to know is the combination of assets that makes up portfolio M in Fig. 2.1 as well as the risk-free asset. This is true for any investor, regardless of his or her degree of risk aversion.

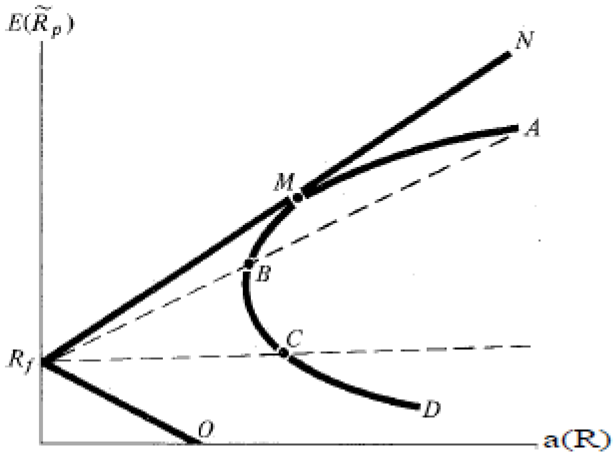


fig 2-1: The efficient set with one risk-free and many risky assets.

- The introduction of a risk-free asset changes the efficient frontier from a curve (for risky assets only) to a straight line (the CAL or CML).
- The efficient frontier now consists of:
    1. Combinations of the risk-free asset and the tangency portfolio (for conservative investors who lend at the risk-free rate).
    2. Leveraged portfolios (for aggressive investors who borrow at the risk-free rate to invest more than 100% in the tangency portfolio).

Figure (2.2) clarifies this point. Investor III is the most risk averse of the three pictured and will choose to invest nearly all of his or her portfolio in the risk-free asset. Investor I, who is the least risk averse, will borrow (at the risk-free rate) toinvest more than 100% of his or her portfolio in the risky portfolio M. However, no investor will choose to invest in any other risky portfolio except portfolio M. For example, all three could attain the minimum variance portfolio at point *B*, but none will choose this alternative because all can do better with some combination of the risk-free asset and portfolio M. Next we shall see that portfolio M can be identified as the market portfolio of all risky assets. All risky assets are held as part of risky portfolio M.

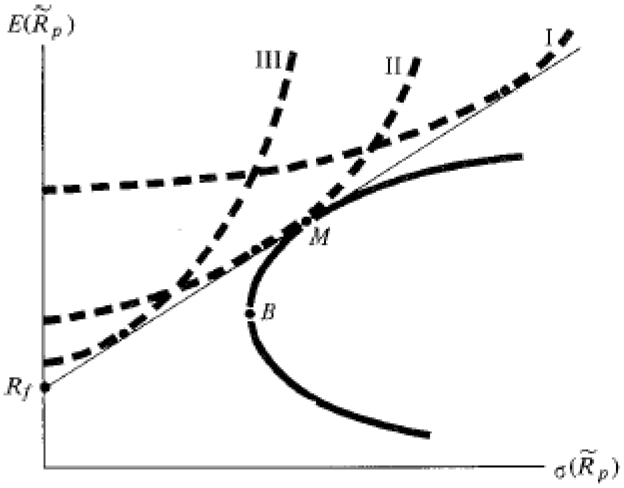


fig 2.2: Dominance of the linear efficient set.

The opportunity set with *N* risky assets and one risk-free asset allows investors to achieve better risk-return trade-offs by combining the risk-free asset with the optimal risky portfolio (the tangency portfolio). The efficient frontier becomes a straight line (the CAL or CML), and investors can choose their desired level of risk by adjusting the proportion of their investment in the risk-free asset and the tangency portfolio. This framework is the foundation of the **Capital Asset Pricing Model (CAPM)** and modern portfolio theory.

2. Capital Market Equilibrium with Risk-Free Asset

Imagine a marketplace where all investors have access to a risk-free savings account (like a government bond) and a stock market index fund (representing the market portfolio). Each investor can decide how much to invest in the risk-free asset (lending) and how much to invest in the index fund (buying risky assets). Some investors might even borrow at the risk-free rate to invest more in the index fund, increasing their potential returns (and risks).

In this scenario:

- The **risk-free rate** acts as a baseline return for all investors.
- The **market portfolio** represents the optimal risky investment, as it is diversified and captures the market's overall risk-return trade-off.
- The **Capital Market Line** shows the best possible trade-off between risk and return, given the risk-free rate and the market portfolio.

For the market to be in equilibrium, we require a set of market-clearing prices. All assets must be held. In other words the existence of equilibrium requires that all prices be adjusted so that the excess demand for any asset will be zero. This market clearing condition implies that an equilibrium is not attained until the single-tangency portfolio, M, which all investors (with homogeneous expectations) try to combine with risk-free borrowing or lending, is a portfolio in which all assets are held according to their market value weights. If *Vi* is the market value of the i th asset, then the percentage of wealth held in each asset is equal to the ratio of the market value of the asset to the market value of all assets. Mathematically:

$$w_j = V_j / \sum_{i=1}^{N} V_i$$

Where wj is the weight of the ith asset in the market portfolio and $\sum Vi$ is the total market value of all assets. Market equilibrium is not reached until the tangency portfolio, M, is the market portfolio. Also, the value of the risk-free rate must be such that aggregate borrowing and lending are equal. The fact that the portfolios of all risk-averse investors will consist of different combinations of only two portfolios is an extremely powerful result. It has come to be known as the two-fund separation principle. Its definition is given below.

3. The Two-Fund Separation Principle

This is a key concept in modern portfolio theory, which states that all investors, regardless of their risk preferences, can construct optimal portfolios by combining just two funds**:**

1. **The Risk-Free Asset**: A risk-free asset (e.g., Treasury bills) with a guaranteed return *Rf*.
2. **The Tangency Portfolio**: The optimal risky portfolio that lies on the efficient frontier and is tangent to the Capital Allocation Line (CAL).

This principle simplifies portfolio selection by showing that investors only need to decide how to allocate their wealth between these two funds, rather than selecting individual risky assets. The separation arises because the tangency portfolio is the same for all investors, and individual risk preferences only determine the proportion of wealth allocated to the risk-free asset versus the tangency portfolio.

For this principle to hold, **the following hypotheses** need to be verified:

- End of period wealth follows normal distribution
- Borrowing and lending rates (rb=rl) are equal to the market risk free rate
- Individuals have homogenous beliefs about probability distributions of rates of return, therefore perceiving the same efficient set
- Market is in equilibrium: excess demand and supply for all assets is zero, aggregate demand = aggregate supply, assets in market portfolio are held according to their market value weight.

4. The Capital Market Line

This setup ensures that all investors, regardless of their risk preferences, use the same market-determined price of risk to make their investment decisions, leading to a single equilibrium in the market. This is very similar to the original description, but framed within the context of the CAPM and the Capital Market Line.

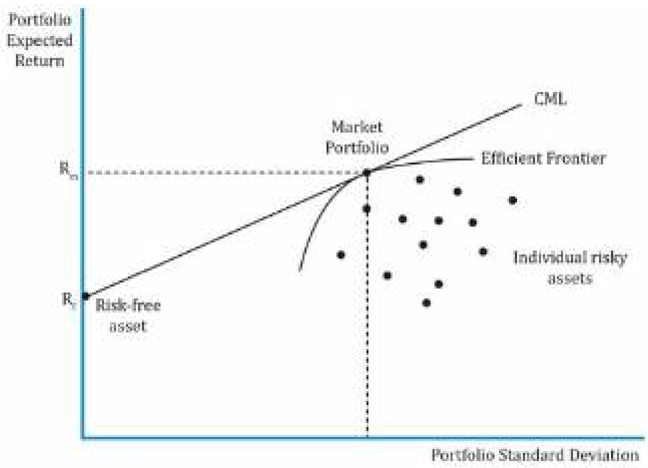


fig (2.3): The Capital Market Line

5. The Unifying Role of the Capital Market Line

a) **Portfolio Return**: The return of the combined portfolio is:

$$E(Rp)=w\cdot E(R_{\text{tangency}})+(1-w)\cdot Rf$$

where:

- w = weight in the tangency portfolio,
- $1-w$ = weight in the risk-free asset,

- $E(R\text{tangency})$ = expected return of the tangency portfolio.

b) **Portfolio Risk**:

- The risk of the combined portfolio is:$\sigma p = w \cdot \sigma\text{tangency}$

  where σtangency is the standard deviation of the tangency portfolio.

c) **Capital Market Line (CML)**:

- The CAL represents all possible combinations of the risk-free asset and the tangency portfolio:

$$E(R_p) = R_f + \frac{E(R_{\text{tangency}}) - R_f}{\sigma_{\text{tangency}}} \cdot \sigma_p$$

- The slope of the CML is the **Sharpe ratio** of the tangency portfolio.

Having established the principle of two-fund separation and defined the capital market line, we find it useful to describe the importance of capital market equilibrium from an individual's point of view.

We wish to compare expected utility-maximizing choices in a world without capital markets with those in a world with capital markets (Fig. 2.4). a capital market is nothing more than the opportunity to borrow and lend at the risk-free rate. As we know, In a world with certainty everyone was better off, given that capital markets existed and Fisher separation obtained. Now we have extended this result to a world with mean-variance uncertainty. Everyone is better off with capital markets where two-fund separation obtains.

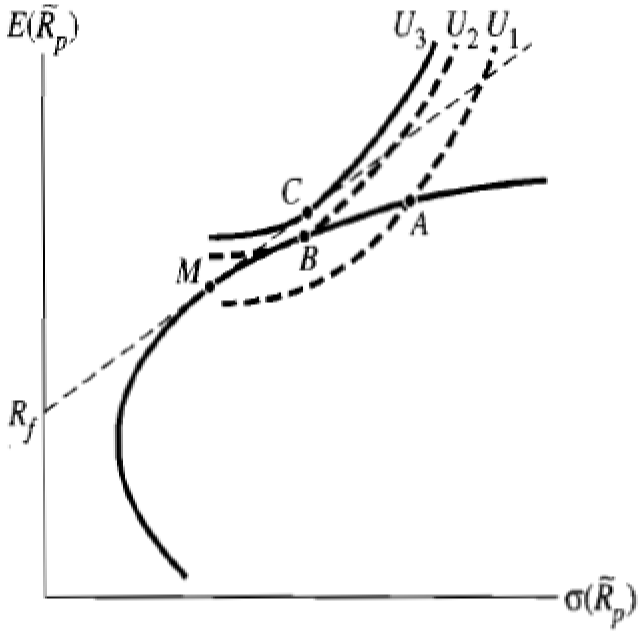


FIG 2.4 : Individual expected utility maximization in a world with capital markets.

Figure (2.4) shows us endowed with the mean-variance combination at point A. With a capital market, we always have two choices available. We can move along the mean-variance opportunity set (by changing our portfolio of risky assets), or we can move along the Capital Market Line by borrowing or lending.

Initially, at point *A,* the trade-off between return and risk is more favorable along the opportunity set than along the market line. Therefore we will move along the opportunity set toward point *B* where the marginal rate of transformation between return and risk on the opportunity set is equal to our subjective marginal rate of substitution along our indifference curve. In the absence of capital markets, we would have maximized our expected utility at point *B.* This would be the Robinson Crusoe solution, and our level of utility would have increased from *U1* to *U2.* However, if we have the opportunity to move along the Capital Market Line, we can be even better off. By moving to point M, then borrowing to reach point *C,* we can increase our expected utility from *U2* to U3.

Therefore we have three important results. First, nearly everyone is better off in a world with capital markets (and no one is worse off). Second, two-fund separation obtains. This means that everyone, regardless of the shape of his or her indifference curve, will decide to hold various combinations of two funds: the market portfolio and the risk-free asset. And third, in equilibrium, the marginal rate of substitution (MRS) between return and risk is the same for all individuals, regardless of their subjective attitudes toward risk.

If the marginal rate of substitution between risk and return is the same for every individual in equilibrium, then the slope of the Capital Market Line is the equilibrium price of risk (EPR):

$$MPR = MRS_{\sigma}^{\mu_p} = (E(R_m) - R_f)/\sigma_m$$

The implication is that decision makers, e.g., managers of firms, can use the market determined equilibrium price of risk to evaluate investment projects regardless of the tastes of shareholders. Every shareholder will unanimously agree on the price of risk even though different shareholders have different degrees of risk aversion. Also the marginal rates of substitution between risk and return for the ith and jth individuals in equilibrium will equal the marginal rate of transformation, and both will be equal to the equilibrium price of risk.

$$\mathrm{MRS}_i = \mathrm{MRS}_j = \frac{E(R_m) - R_f}{\sigma(R_m)} = \mathrm{MRT}.$$

Example: Mohamed is constructing a portfolio using T-bills and an index fund that tracks the KSE 100 Index. The T-bills have a return of 10%, and the index fund has an expected return of 16% with a standard deviation of 30%.

1. Draw the Capital Market Line (CML).
2. Identify the point on the CML where the allocation to the index fund is 75%.
3. Calculate the expected return and risk (standard deviation) at this point.

Solution

1. Identify the Two Assets for the CML

- **Risk-Free Asset (Rf):** T-bills with a return = **10%**
- **Risky Market Portfolio (M):** The KSE 100 Index Fund with:

  - Expected Return (E(Rm)) = **16%**
  - Standard Deviation (σm) = **30%**

The Capital Market Line (CML) is the straight line drawn from the risk-free rate on the Y-axis (0% risk) through the market portfolio (M) at (30% risk, 16% return).

2. Find the Portfolio where Investment in the Market is 75%

This means the portfolio weights are:

- Weight in Market (w_m) = **0.75**
- Weight in T-bills (w_f) = **0.25**

3. Calculate the Expected Return at this Point

The expected return of the portfolio E(Rp) is the weighted average:
E(Rp) = (w_f × Rf) + (w_m × E(Rm))

E(Rp) = (0.25 × 10%) + (0.75 × 16%)

E(Rp) = 2.5% + 12%   **E(Rp) = 14.5%**

4. Calculate the Risk (Standard Deviation) at this Point

Because T-bills are risk-free, the portfolio's standard deviation (σp) is simply the weight in the risky asset multiplied by its standard deviation:
σp = w_m × σm

σp = 0.75 × 30%   **σp = 22.5%**

The point on the CML where the investment in the market index is 75% has an:

- **Expected Return = 14.5%**
- **Standard Deviation (Risk) = 22.5%**

This point will lie on the straight CML between the risk-free asset (10% return, 0% risk) and the market portfolio (16% return, 30% risk).

6. Diagonal model and Market model

The Diagonal Model and the Market Model are both financial models used to analyze and predict stock returns, but they have different structures and applications. Here's a breakdown of each:

6.1. Diagonal Model (or Diagonal Index Model)

The Diagonal Model, introduced by (Sharpe, 1964), assumes that stock returns are linearly related to a single market index (e.g., S&P 500). It simplifies the covariance structure by assuming that stock-specific returns are uncorrelated.

The return of stock *i* is given by: $R_{i,t}=\alpha_i+\beta_i R_{m,t}+\epsilon_{i,t}$

where:

- $R_{i,t}$ = Return of stock *i* at time *t*,
- α_i = Abnormal return (independent of the market),
- β_i = Sensitivity to the market return,
- $R_{m,t}$ = Market return at time *t*,
- $\epsilon_{i,t}$ = Idiosyncratic error term ($E[\epsilon_{i,t}]=0$, Cov $(\epsilon_{i,t},R_{m,t})=0$).

Assomptions:

- Assumes no covariance between individual stocks except through the market.
- Reduces computational complexity compared to the full Markowitz model.
- Used in portfolio optimization to estimate risk and return efficiently.

6.2. Market Model

The Market Model is a statistical model that relates stock returns to market returns, often used in event studies and the Capital Asset Pricing Model (CAPM)**.**

The model is identical in form to the Diagonal Model: $R_{i,t}=\alpha_i+\beta_i R_{m,t}+\varepsilon_{i,t}$

- ✓ Only one source of correlation**:** the market.
- ✓ Residual terms ε_i are assumed to be **uncorrelated** across assets

The following table (1.1) provides a detailed comparison of the two models across several important dimensions.

Table 1.1: Comparison of the Diagonal Model and Market Model

Feature	**Diagonal Model**	**Market Model**
Correlation	Assumes no correlation	Correlation comes only from market
Covariance structure	Off-diagonal elements = 0	Covariance driven by β and market var
Risk components	Only variance (no covariances)	Systematic + Idiosyncratic
Complexity	Very low	Moderate
Realism	Low	Higher

7. Benefits of diversification

Portfolio diversification is a technique that reduces the impact of individual asset risk on the overall portfolio. It is based on the premise that different assets will perform differently under various market conditions, and by combining them, an investor can achieve a more stable and potentially more profitable investment experience.

To study the difference between portfolio risk and individual asset risk, some data was collected (see pp 137-) and the following results were obtained:
i) Bayside smoke asset: average monthly return: 0,45%, average yearly return: 5,4%, standard deviation: 7,26%
ii) 100-stock portfolio average monthly return: 0,91%, average yearly return: 10,9%, standard deviation: 4,45%
You would expect: a) a well diversified portfolio would have lower risk and b) a riskier asset would have higher return.
We see that the first proposition is coherent with the above results while the second is not. Is there an explanation for this apparent paradox?

The first proposition is related to the question: What happens when the number of assets in the portfolio increases?

We begin by taking a look at what happens to portfolio variance as we increase the number of assets in a portfolio.

$$\text{VAR}(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij},$$

In general the number of variance terms equals the number of assets in the portfolio, *N*, whereas the number of covariance terms equals $(N^2 — N)$ or $N(N — 1)$ .Suppose that we have an equally weighted portfolio so that $w_i = w_j = 1/N$. Then the portfolio variance can be written:

$$\text{VAR}(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{N}\frac{1}{N}\sigma_{ij} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}.$$

This expression can be separated into variance and covariance terms as follows:

$$\text{VAR}(R_p) = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_{ii} + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}\sigma_{ij} \qquad (1)$$

Suppose that the largest individual asset variance is L. Then the first term, the variance term, is always less than or equal to:

$$\frac{1}{N^2}\sum_{i=1}^{N} L = \frac{LN}{N^2} = \frac{L}{N}$$

and as the number of assets in the portfolio becomes large, this term approaches zero:

$$\lim_{N\to\infty}\frac{L}{N} = 0.$$

On the other hand, the covariance terms do not vanish. Let $\bar{\sigma}_{ij} = \sum_i^N \sum_{j\neq i}^N \sigma_{ij} / N(N-1)$ be the average covariance. Then in the right-hand term in Eq. (1), there are $(N^2 — N)$ covariance terms, all equal to $\bar{\sigma}_{ij}$; therefore the right-hand term can be rewritten as:

$$\frac{1}{N^2}(N^2 - N)\bar{\sigma}_{ij} = \frac{N^2}{N^2}\bar{\sigma}_{ij} - \frac{N}{N^2}\bar{\sigma}_{ij},$$

and the limit as *N* approaches infinity is: $\lim_{N\to\infty}\left(\frac{N^2}{N^2}\bar{\sigma}_{ij} - \frac{N}{N^2}\bar{\sigma}_{ij}\right) = \bar{\sigma}_{ij}$

Consequently, as we form portfolios that have large numbers of assets and that are better diversified, the covariance terms become relatively more important.
In the next chapter, it will be shown that, given an equilibrium market setting, only the portion of total variance which is correlated to the economy is relevant. Any portion of the total risk which is not correlated with the economy can be avoided at no cost through diversification.

The second proposition can be dealt with by evaluating the contribution of a single asset to the total variance of a portfolio by taking the partial derivative of this portfolio variance with respect to the proportion invested in the asset i:

$$\partial V(R_p)/\partial x_i = \partial(\sum\sum x_i\, x_j \sigma_{ij})/\partial x_i = \partial(\sum x_i^2\, \sigma_{ii} + \sum\sum x_i\, x_j \sigma_{ij})/\partial x_i$$

$$\frac{\partial V(R_p)}{\partial x_i} = 2x_i\, \sigma_{ii} + 2\sum x_j \sigma_{ij} = \frac{2\sigma_{ii}}{N} + 2\sum \sigma_{ij}\, /N$$

Here also, we are assuming an equally diversified portfolio, and then:

$$V(Rp)/\partial x_i = \frac{2\sigma_{ii}}{N} + \frac{2\sum \sigma_{ij}}{N} = \frac{2\sigma_{ii}}{N} + \left(\frac{2}{N}\right)(N-1)\overline{\sigma_{\iota.}} =$$

$$\lim_{N\to\infty} V(Rp)/\partial x_i = (2)\overline{\sigma_{\iota.}}$$

We see that, the covariance plays an important role in the expression of risk. We said in our course notes that economic agents are interested in two dimensions: the average rate of return and the variance, but here we show that it is rather the covariance which expresses the risky nature of an asset. Of course, we use intuition to present the role of covariance as a measure of the asset risk; but we will use a more rigorous approach in our next lesson to show that it is the covariance of the asset which indicates its risky nature.

8. Short-sale constraints and numerical solutions

In modern portfolio theory, investors are often assumed to have the ability to short-sell assets—that is, sell securities they do not own, expecting to buy them back later at a lower price. This assumption allows for more flexible portfolio construction and can lead to negative asset weights in optimization models. However, in many real-world situations, short-selling is restricted or prohibited due to regulatory limits, borrowing costs, or institutional policy. These limitations are known as short-sale constraints.

When short sales are not allowed, the optimization problem must be modified to ensure that all asset weights remain non-negative: $w_i \geq 0$ for all i

This transforms the classical mean-variance optimization into a constrained quadratic programming problem, where the solution must lie within the non-negative orthant of the weight space. As a result, the efficient frontier under short-sale constraints is typically concave and lies below the unconstrained frontier, indicating reduced diversification and higher risk for the same level of expected return.

From a numerical standpoint, solving this constrained problem requires specialized algorithms such as:

- **Quadratic programming solvers**
- **Interior-point methods**
- **Active-set methods**

These algorithms handle the nonlinear and inequality-bound nature of the problem, providing feasible portfolios that comply with short-sale restrictions. In practice, the imposition of such constraints often results in corner solutions, where some asset weights are exactly zero—reflecting assets that would have been shorted in the unconstrained case.

In summary, while short-sale constraints bring models closer to practical reality, they also increase the computational complexity and limit the investor's ability to fully diversify, which can affect overall portfolio performance.

Exercises

Exercise1

Explain in your own words how the inclusion of a risk-free asset transforms the efficient frontier from a curve to a straight line.

- What is the economic intuition behind this?
- Why will all investors choose a combination of the risk-free asset and the tangency portfolio?

Exercise 2

An investor can allocate funds between:

- A risk-free asset yielding 5% per year
- A risky portfolio with $E(R_m)=12\%$ and $\sigma_m=20\%$

Tasks:

1. Write the equation of the Capital Market Line (CML).
2. Compute the expected return and standard deviation of a portfolio that invests 60% in the risky portfolio and 40% in the risk-free asset.
3. Interpret the results.

Exercise 3

Explain why the two-fund separation principle implies that portfolio managers can design a single risky portfolio for all investors, even though their levels of risk aversion differ.
What practical advantage does this create for financial institutions?

Exercise 4

Given two assets A and B with the following characteristics:

Asset	Expected Return	Standard Deviation	Correlation (A,B)
A	10%	12%	0.4
B	8%	10%	—

If an investor invests 50% in each, compute:

1. Portfolio variance and standard deviation.
2. Discuss how correlation affects diversification.

Exercise 5

Explain how short-sale constraints modify the efficient frontier. What does it mean for an investor when the optimal portfolio under no constraints has negative weights for some assets?

Chapter 3: Financial Market Equilibrium: The Capital Asset Pricing Model (CAPM)

Much of this chapter focuses on extending the concept of market equilibrium to derive the market price of risk and the appropriate risk measure for an individual asset. A foundational economic model addressing this problem was developed independently by sharpe and (Treynor, 1961), with later contributions from (Mossin, 1966), (Lintner, 1965), and (Black, 1972). The first model we examine is the Capital Asset Pricing Model (CAPM)**,** it demonstrates that the equilibrium returns of risky assets depend on their covariance with the market portfolio.

In financial market equilibrium, all assets are priced such that no investor can achieve a higher expected return for a given level of risk. Under these conditions, all investors hold some combination of the market portfolio and the risk-free asset. The equilibrium expected return on each asset reflects its systematic contribution to the overall market risk.

The Capital Asset Pricing Model (CAPM) formalizes this equilibrium by linking each asset's required return to its covariance with the market portfolio, establishing a market-consistent pricing mechanism for risk.

A useful way to visualize this equilibrium is with a diagram showing the demand for risky assets (driven by investor risk aversion) and the fixed aggregate supply of risky assets. The intersection point represents the equilibrium expected return on the market.

1- **CAPM Assumptions**

The CAPM is developed in a hypothetical world where the following assumptions are made about investors and the opportunity set:

1. Investors are risk-averse individuals who maximize the expected utility of their end-of-period wealth.
2. Investors are price takers and have homogeneous expectations about asset returns that have a joint normal distribution.
3. There exists a risk-free asset such that investors may borrow or lend unlimited amounts at the risk-free rate.
4. The quantities of assets are fixed. Also, all assets are marketable and perfectly divisible.

5. Asset markets are frictionless and information is costless and simultaneously available to all investors.
6. There are no market imperfections such as taxes, regulations, or restrictions on short selling.

The Capital Asset Pricing Model (CAPM) states that, in equilibrium, the expected return of an asset compensates only for its systematic risk (market risk), measured by its beta (β). This is because:

- Unsystematic (diversifiable) risk can be eliminated by holding a well-diversified portfolio.
- Investors are only rewarded for taking on non-diversifiable (market) risk.

The expected return of asset *j* is given by:

$$\mu j = Rf + \beta j(\mu m - Rf)$$

where:

- μj = Expected return of asset *j*,
- Rf = Risk-free rate (e.g., government bond yield),
- μm = Expected return of the market portfolio,
- βj = Sensitivity of asset *j* to market movements, defined as:βj=var(Rm)cov(Rj,Rm)

2. Systematic and Nonsystematic Risk

Investment risk is conventionally categorized into two primary types: systematic and nonsystematic risk. This distinction is fundamental to modern portfolio theory and asset pricing.

2.1 Systematic Risk, also termed market risk or non-diversifiable risk, refers to the component of risk that is inherent to the entire market or economic system. This risk factor cannot be eliminated through diversification, as it affects a broad range of assets. Consequently, investors are compensated with an expected return for bearing systematic risk. Prevalent sources of systematic risk include fluctuations in interest rates, inflationary pressures, major geopolitical events, and widespread natural disasters. While its impact cannot be entirely avoided, its effect on a specific portfolio can be mitigated by selecting securities with low betas or low correlation to the broader market.

2.2 Nonsystematic Risk, in contrast, is risk that is specific to an individual asset, company, or industry. Also known as diversifiable risk or specific risk, it arises from factors such as firm-specific management decisions, product failures, regulatory changes affecting a single sector, or competitive pressures. Since these risks are unique to individual assets, they can be effectively diversified away by constructing a portfolio containing a large number of unrelated assets across various industries and asset classes. In an efficient market, no risk premium is awarded for bearing nonsystematic risk, as it is considered avoidable.

2.3 The Pricing of Risk: The process of pricing a risky asset is functionally equivalent to determining its expected rate of return, which must compensate investors solely for its systematic risk exposure. This is the foundational principle of models such as the Capital Asset Pricing Model (CAPM). The asset's price is calculated by discounting its expected future cash flows at a rate that reflects this required return, thereby formally linking an asset's valuation to its non-diversifiable risk profile.

Example 1: Describe the systematic and nonsystematic risk components of the following assets:

1. A six-month U.S. Treasury bill.
2. An index fund designed to track the S&P 500, which exhibits a total risk (standard deviation) of 18%.

Solution:

1. **A six-month Treasury bill:** A Treasury bill is a risk-free asset in nominal terms for an investor with a matching six-month horizon. The payment at maturity is known with certainty. Consequently, it carries neither systematic nor nonsystematic risk, resulting in a total variance of zero.
2. **An S&P 500 Index Fund:** The S&P 500 is a proxy for the market portfolio. By definition, a market portfolio is fully diversified and contains only systematic risk. Therefore, the fund's total risk of 18% is attributable entirely to systematic risk, with a nonsystematic risk component of zero.

Example 2

Consider two assets, X and Y. Asset X has a total risk of 25%, of which 15% is systematic risk. Asset Y is the S&P 500 index fund from the previous example, with a total risk of 18%, all of which is systematic. Which asset will have a higher expected rate of return?

Solution:

A comparison based solely on total risk would incorrectly suggest Asset X should offer a higher return. However, according to asset pricing theory, expected return is solely a function of systematic risk, as nonsystematic risk can be eliminated through diversification and therefore is not priced.

- Asset X has a systematic risk of 15%.
- Asset Y has a systematic risk of 18%.

Since Asset Y has a higher level of systematic risk, it will command a higher expected rate of return. Investors are not compensated for bearing the diversifiable, nonsystematic risk present in Asset X.

3-The efficiency of the market portfolio

For the Capital Asset Pricing Model (CAPM) to hold, the market portfolio must be efficient—meaning it must lie on the upper half of the minimum-variance opportunity **set** (the efficient frontier, as illustrated in Fig.3.1)

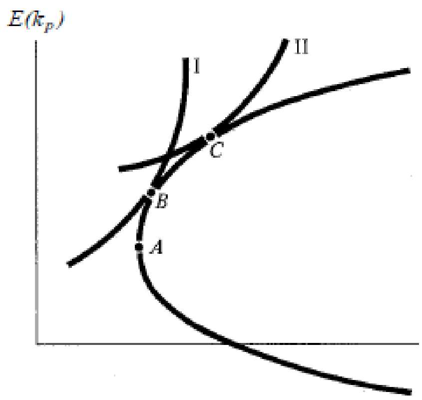


Figure 3.1: All investors select efficient portfolios

1. Even without a risk-free asset, all investors will select efficient portfolios (i.e., portfolios offering the highest expected return for a given level of risk).
2. Since investors may differ in risk tolerance, their individual portfolios will vary along the efficient frontier.
3. However, when aggregated, the market portfolio—which is the weighted average of all individual portfolios—must also be efficient.

3- Derivation Of The CAPM

The Capital Asset Pricing Model **(CAPM)** is derived from Markowitz's portfolio theory and relies on several key assumptions. Its fundamental meaning is to establish a linear relationship between an asset's expected return and its systematic risk (measured by beta) in an equilibrium market.

1- In the equilibrium, there is no excess demand and prices must adjust so that demand of an asset is equal to its supply and consequently the market portfolio consists of all marketable assets held in the proportion of their market value weight:

Xi= Market value of asset i / Total market value of all assets.

2- The portfolio, consisting of a% invested in the risky asset I and (1-a)% invested in market portfolio, has the following parameters:

$$\hat{R}_p = a\,I + (1-a)\,M \quad (1)$$

$$E(R_p) = a\,E(I) + (1-a)\,E(M) \quad \text{or } \mu_p = a\,\mu_I + (1-a)\,\mu_M \quad (1)$$

$$\text{Or } \sigma^2(\hat{R}_p) = a^2\,\sigma^2(I) + (1-a)^2\,\sigma^2(M) + 2\,a\,(1-a)\,\operatorname{cov}(I, M) \quad (2)$$

$$\text{Or } \sqrt{\sigma^2(\hat{R}_p)} = \sigma_p = (a^2\,\sigma_I^2 + (1-a)^2\,\sigma_M^2 + 2\,a\,(1-a)\,\sigma_{IM})^{1/2} \quad (3)$$

3- To get the change in the expected rate of return (μp) and the standard deviation σp of the portfolio with respect to a (the proportion of investment in the risky asset I), derivatives are computed as follows:

$$\frac{\partial E(\tilde{R}_p)}{\partial a} = E(\tilde{R}_i) - E(\tilde{R}_m),$$

$$\frac{\partial \sigma(\tilde{R}_p)}{\partial a} = \frac{1}{2}\left[a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}\right]^{-1/2} \times \left[2a\sigma_i^2 - 2\sigma_m^2 + 2a\sigma_m^2 + 2\sigma_{im} - 4a\sigma_{im}\right].$$

4- "a" represents the excess demand for asset I (the "normal" demand is already included in the market portfolio). But because, this is an excess demand in the equilibrium situation, it has to be null. Therefore, the change in the expected rate of return (μp) and the standard deviation σp of the portfolio with respect to "a" must be studied when a=0:

$$\left.\frac{\partial E(\tilde{R}_p)}{\partial a}\right|_{a=0} = E(\tilde{R}_i) - E(\tilde{R}_m),$$

$$\left.\frac{\partial \sigma(\tilde{R}_p)}{\partial a}\right|_{a=0} = \frac{1}{2}(\sigma_m^2)^{-1/2}(-2\sigma_m^2 + 2\sigma_{im}) = \frac{\sigma_{im} - \sigma_m^2}{\sigma_m}.$$

The slope of the risk-return trade-off evaluated at point M, in market equilibrium, is:

$$\left.\frac{\partial E(\tilde{R}_p)/\partial a}{\partial \sigma(\tilde{R}_p)/\partial a}\right|_{a=0} = \frac{E(\tilde{R}_i) - E(\tilde{R}_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}.$$

5- The CML is an equilibrium outcome, thus the tradeoff between the expected rate of return and the risk of a portfolio is equal to the slope of the CML: $\frac{E(\tilde{R}_m) - R_f}{\sigma_m}$, where σ_m is the standard deviation of the market portfolio. Equating this with the slope of the opportunity set at point M we have $\frac{E(\tilde{R}_m) - R_f}{\sigma_m} = \frac{E(\tilde{R}_i) - E(\tilde{R}_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}$.

This relationship can be arranged to solve for $E(\tilde{R}_i)$ as follows:

$$E(\tilde{R}_i) = R_f + \left[E(\tilde{R}_m) - R_f\right]\frac{\sigma_{im}}{\sigma_m^2}.$$

This equation is known as the capital asset pricing model, CAPM. it is also called the security market line (SML). The required rate of return on any asset is equal to the risk-free rate of return plus a risk premium. The risk premium is the price of risk multiplied by the quantity of risk. In the terminology of the CAPM, the price of risk is the slope of the line, the difference between the expected rate of return on the market portfolio and the riskfree rate of return. The quantity of risk is often called beta, $\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\mathrm{COV}(\tilde{R}_i, \tilde{R}_m)}{\mathrm{VAR}(\tilde{R}_m)}$.

Note that : $E(\tilde{R}_m) - R_f$ represents the price of risk or the market risk premium

β_i represents the quantity of risk (or level of risk). While their product is called the risk premium.

4/ Properties of the CAPM model

There are several properties of the CAPM that are important.

1- In equilibrium, every asset must be priced so that its risk-adjusted required rate of return falls exactly on the SML.

Investors can always diversify away all risk except the covariance of an asset with the market portfolio. In other words, they can diversify away all risk except the risk of the economy as a whole, which is inescapable (undiversifiable).

Consequently, the only risk that investors will pay a premium to avoid is covariance risk. The total risk of any individual asset can be partitioned into two parts—systematic risk, which is a measure of how the asset covaries with the economy, and unsystematic risk, which is independent of the economy.

Total risk = systematic risk + unsystematic risk,

Empirically, the return on any asset is a linear function of market return plus a random error term which is independent of the market:

$$\tilde{R}_j = a_j + b_j \tilde{R}_m + \tilde{\varepsilon}_j.$$

We can write the variance of this relationship as:

$$\sigma_j^2 = b_j^2 \sigma_m^2 + \sigma_\varepsilon^2.$$ **Called the diagonal or the market model**

From linear analysis, it can be shown that : $b_j = \sigma_{jm} / \sigma_m^2$

The assumptions used for this simple regression model are stated below:

$$\text{h1}: \; E(\varepsilon_{jt}) = 0 \;\; \text{and} \;\; V(\varepsilon jt) = E(\varepsilon jt^2) = \sigma j^2 \;, \forall t$$

$$\text{h2}: \; Cov(\varepsilon jta, \varepsilon jtb) = E(\varepsilon jta.\varepsilon jtb) = 0, \forall ta \neq tb$$

$$\text{h3}: \; Cov(\varepsilon jt, Rmt) = E(\varepsilon jt.Rmt) = 0, \forall t$$

2- A second important property of the CAPM is that the measure of risk for individual assets is linearly additive when the assets are combined into portfolios. For example, if we put *a%* of our wealth into asset *X,* with systematic risk of βx, and *b%* of our wealth into asset Y, with systematic risk of *βy,* then the beta of the resulting portfolio, βp, is simply the weighted average of the betas of the individual securities: $\beta_p = a\beta_x + b\beta_y$.(see copeland pp154-155).

The fact that portfolio betas are linearly weighted combinations of individual asset betas is an extremely useful tool. All that is needed to measure the systematic risk of portfolios is the betas of the individual assets.

Note that ,at the margin, the change in the contribution of asset *i* to portfolio risk is simply: $\mathrm{COV}(R_i, R_p)$.

Covariance risk (the covariance between an asset's returns and the portfolio's returns) is the true measure of risk because:

- It quantifies how adding an asset changes the overall portfolio risk (variance).
- An asset’s contribution to portfolio risk depends not just on its own volatility, but on how it co-varies with other assets.
- Diversification benefits arise when assets have low or negative covariance, reducing total portfolio risk.

While CAPM uses beta (β)**,** which is derived from covariance, to represent systematic risk, these terms can be misleading because:

- They assume costless, perfect diversification (i.e., investors can eliminate all idiosyncratic risk).
- They rely on the existence of a large, well-diversified market portfolio (which may not exist in practice).
- In smaller or less efficient markets, covariance risk remains relevant even if full diversification isn’t possible**.**

5. Characteristics of the CAPM

- **Simplicity and Intuition:** The model is easy to use and understand, offering a clear link between risk (via beta) and return.

- **Linear Relationship:** There is a direct linear relationship between expected return and beta.
- **Benchmarking Tool:** CAPM is widely used to estimate the cost of equity and to evaluate asset pricing and portfolio performance.
- **Theoretical Foundations:** Rooted in equilibrium theory, linking asset pricing to investor behavior.

However, the model is also subject to criticisms due to its strong assumptions and empirical limitations.

6. The Security Market Line (SML)

The **Security Market Line (SML)** is a graphical representation of the CAPM. It plots the expected return of an asset as a function of its beta:

- **X-axis (β):** Measures systematic risk (non-diversifiable).
- **Y-axis (μ):** Measures expected return.

6.1. Equation of the SML:

$\mu = Rf + \beta(\mu m - Rf)$

- It is a straight line with:
    - **Intercept** = Rf (return when $\beta=0$, i.e., a risk-free asset).
    - **Slope** = $(\mu m - Rf)$ (market risk premium).

6.2. Interpretation of the SML

- Assets **on** the SML are fairly priced.
- Assets **above** the SML are undervalued (offering higher return for their risk).
- Assets **below** the SML are overvalued.

The slope of the SML represents the market risk premium, and its intercept is the risk-free rate.

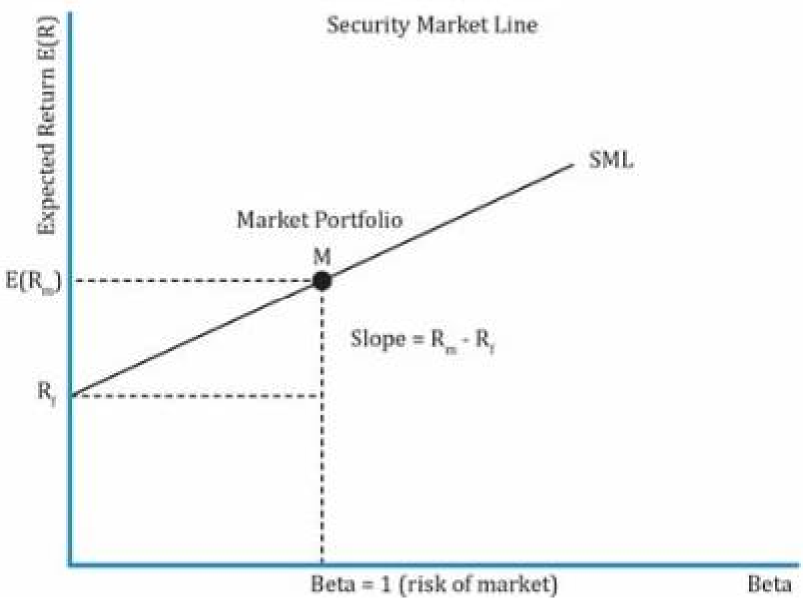


Figure (3.2): SML interpretation

6.3. Differences between the CML and the SML

Basis of Difference	Capital Market Line (CML)	Security Market Line (SML)
Scope of Application	Applies only to efficient portfolios (i.e., those that are perfectly diversified and lie on the efficient frontier).	Applies to any individual security or portfolio, including those that are inefficient.
Risk Measurement	Measures risk using the total risk of the portfolio, represented by the standard deviation (σp) on the x-axis.	Measures risk using systematic risk (non-diversifiable), represented by the beta (β) of the asset or portfolio on the x-axis.
Risk Premium	The risk premium is a function of the portfolio's total risk and the market's Sharpe Ratio.	The risk premium is a function of the asset's systematic risk (beta) and the Market Risk Premium.
Defining Equation	$E(Rp)=Rf+(\sigma mE(Rm)-Rf)\sigma p$	$E(Ri)=Rf+\beta i[E(Rm)-Rf]$
Line Slope	The slope of the CML is the Sharpe Ratio of the market portfolio: $E(Rm)-Rf / \sigma m$.	The slope of the SML is the Market Risk Premium:

Example

David allocates his portfolio as follows: 15% to the risk-free asset, 45% to a market-tracking index fund, and 40% to a high-risk stock with a beta of 1.8. The risk-free rate is 4% and the expected market return is 11%. What is the portfolio beta and expected return?

Solution:

- **Portfolio Beta** = Weighted average beta of all assets = $(0.15\times0)+(0.45\times1)+(0.40\times1.8)$

$=1.17(0.15\times0)+(0.45\times1)+(0.40\times1.8)=1.17$

- **Portfolio Return** (using the CAPM/SML formula with the portfolio beta)

$= rf+\beta portfolio(rm-rf)= 0.04+1.17[0.11-0.04]=0.1219$

$=12.19\%0.04+1.17[0.11-0.04]=0.1219=12.19\%$

- **Alternative method:**

First, determine the expected return of the high-risk stock using its beta.
Expected return of high-risk stock = $0.04+1.8[0.11-0.04]=0.166=$

$16.6\%0.04+1.8[0.11-0.04]=0.166=16.6\%$

Portfolio return = Weighted average return of all assets= $(0.15\times4\%)+(0.45\times11\%)$

$+(0.40\times16.6\%) = 12.19\%(0.15\times4\%)+(0.45\times11\%)+(0.40\times16.6\%)=12.19\%$

7. Limitations of the CAPM

7.1. Theoretical Limitations of the CAPM

1. **Single-Factor Model:**

   The CAPM assumes that only systematic or market risk is priced. This assumption is restrictive, as it ignores other potential factors that may influence investment returns, such as firm size, value, momentum, or macroeconomic conditions.

2. **Single-Period Model:**

   The CAPM is built on the assumption that all investors make their decisions over a single investment period. In reality, investors have multi-period horizons, and market conditions evolve over time, making this assumption impractical.

7.2. Practical Limitations of the CAPM

1. **Market Portfolio:**

   The theoretical market portfolio should include all investable assets — not only financial securities but also real assets such as real estate, human capital, and even unique assets like the Eiffel Tower. Since such an all-encompassing portfolio cannot exist in practice, the CAPM's market portfolio remains a theoretical construct.

2. **Proxy for a Market Portfolio:**

   Because a true market portfolio cannot be observed or constructed, analysts typically use proxies such as the S&P 500 or other broad stock indices. However, different analysts may use different proxies, leading to inconsistent estimates. For instance, one analyst might use the S&P 500 for U.S. equities, while another may prefer the DAX for European markets.

3. **Estimation of Beta Risk:**

   Beta is a key parameter in the CAPM, representing the sensitivity of an asset's returns to the market. However, estimating beta is subject to significant error. It depends on the length and frequency of the data used ,for example, daily versus monthly returns, and past beta values may not accurately represent future risk conditions.

4. **CAPM Does Not Predict Returns Accurately:**

   Empirical evidence shows that actual realized returns often deviate from those predicted by the CAPM. This inconsistency suggests that additional factors beyond market risk may influence expected returns.

5. **Homogeneity in Investor Expectations:**

   The CAPM assumes that all investors share identical expectations about returns, variances, and covariances of securities, resulting in one optimal market portfolio. In reality, investors differ in their beliefs, information, and risk preferences, leading to multiple equilibrium portfolios instead of a single one.

Exercises

Exercise 1

You are given the following information about a set of risky assets:

Asset	$E(R_i)$	σ_i	$Cov(R_i, R_m)$
A	0.14	0.25	0.045
B	0.11	0.18	0.027
C	0.08	0.12	0.015

The market has an expected return of 10% and a variance $\sigma^2_m = 0.04$. The risk-free rate is 3%.

1. Compute β_i for each asset.
2. Plot the Security Market Line (SML).
3. Determine whether each asset is overvalued or undervalued.
4. Interpret economically what happens in equilibrium.

Exercise 2

Suppose an investor holds a portfolio consisting of two assets, X and Y:

Asset	Proportion	$E(R_i)$	β_i
X	0.6	0.14	1.2
Y	0.4	0.10	0.8

The risk-free rate is 4%, and the expected market return is 12%.

1. Compute the expected return and beta of the portfolio.
2. Using the CAPM, calculate the equilibrium expected return for a portfolio with that beta.
3. Determine if the portfolio is correctly priced, overvalued, or undervalued.
4. Suggest an arbitrage strategy to restore equilibrium.

Exercise 3

You are given time-series data for three stocks and the market portfolio. Using monthly returns over five years, you estimate the following regression results:

$$R_i - R_f = \alpha_i + \beta_i (R_m - R_f) + \varepsilon_i$$

Asset	α_i	β_i	R^2
D	0.002	1.05	0.92
E	−0.001	0.75	0.88
F	0.004	1.50	0.95

The average market risk premium $(E(R_m) - R_f) = 6\%$.

1. Compute the predicted return for each stock according to the CAPM.
2. Compare predicted vs. realized returns (using α_i) and interpret.
3. Discuss whether CAPM holds for this dataset.
4. Suggest an alternative model (for example, APT or Fama-French) that might better capture the observed deviations.

Exercise 4

Consider a continuous compounding environment with a risk-free rate of 3%, an expected market return of 9%, and volatility of the market $\sigma_m = 20\%$.

1. Express the CAPM in continuous-time form.
2. If an asset has $\beta = 1.3$, compute its expected continuously compounded return.
3. Compute the equivalent discrete annualized rate.
4. Discuss how the continuous formulation changes risk pricing in high-frequency trading contexts.

Exercise 5

Derive the CAPM starting from the mean-variance optimization problem under the following assumptions:

- All investors maximize $E(R_p) - (\lambda/2)\sigma_p^2$.
- All investors can borrow and lend at the risk-free rate R_f.
- There exists a market portfolio M containing all risky assets.

1. Show that the tangency condition implies: $E(R_i) - R_f = \beta_i [E(R_m) - R_f]$.

2. Provide a clear, step-by-step derivation, identifying all mathematical assumptions used.

Chapter 4: Empirical Validation of CAPM and extensions

The Capital Asset Pricing Model (CAPM) holds significant utility beyond its theoretical contributions, serving as a critical tool for financial decision-making. Its primary practical application lies in the estimation of expected returns, which forms the basis for security valuation and capital budgeting.

By providing a required rate of return that is directly tied to an asset's systematic risk (beta), the CAPM offers an objective discount rate. In security valuation, this CAPM-derived rate is used to discount the expected future cash flows from an asset—such as dividends or a terminal sale price—to arrive at its fundamental intrinsic value. Simultaneously, in corporate finance, the model is indispensable for capital budgeting. Here, the CAPM helps determine a project-specific cost of equity, which serves as the hurdle rate. A project is considered economically feasible and value-accretive to the firm only if its expected return exceeds this risk-adjusted benchmark, ensuring that shareholders are adequately compensated for the systematic risk undertaken.

1. The Market Model vs. the Capital Asset Pricing Model (CAPM)

In modern portfolio theory, both the Market Model and the Capital Asset Pricing Model **(CAPM)** describe the relationship between an individual asset's return and the return of the overall market. Although they are mathematically related, their objectives, assumptions, and interpretations differ significantly.

Aspect	Market Model	CAPM
Nature	Empirical (statistical)	Theoretical (equilibrium-based)
Equation	$R_i = \alpha_i + \beta_i R_m + \varepsilon_i$	$E(R_i) = R_f + \beta_i [E(R_m) - R_f]$
Objective	Estimate realized returns based on market data	Explain expected returns under equilibrium
Intercept (α)	Measures abnormal performance	Should be zero if CAPM holds
β (Beta)	Estimated from regression	Represents systematic risk
Error term (ε)	Captures unsystematic risk	Not present (model is theoretical)
Use	Empirical testing, event studies	Asset pricing, cost of equity, valuation

1.1. Link between the Models

The CAPM can be seen as the theoretical foundation of the Market Model. If the CAPM holds true, then in the Market Model:

$$E(\alpha i)=0 \quad \text{and} \quad E(Ri)=Rf+\beta i[E(Rm)-Rf]$$

Thus:

- The Market Model tests the empirical validity of CAPM.
- A statistically significant **α** (alpha) indicates abnormal performance not explained by market risk — evidence against the CAPM.

1.2. Applications in Practice

- **Market Model:**
  - Used in event studies to isolate abnormal returns around corporate announcements.
  - Applied for risk decomposition (systematic vs. unsystematic).
- **CAPM:**
  - Used to estimate the cost of equity capital in corporate finance.
  - Serves as a benchmark for portfolio performance evaluation.
  - Forms the basis for multi-factor models like the Fama-French 3-factor model.

2. The Use of the CAPM model for valuation: single period with uncertainty

Because it provides a quantifiable measure of risk for individual assets, the CAPM is an extremely useful tool for valuing risky assets. For the time being, let us assume that we are dealing with a single time period. This assumption was built into the derivation of the CAPM.

We want to value an asset that has a risky payoff at the end of the period. Call this $\widetilde{R}_j$. It could represent the capital gain on a common stock or the capital gain plus a dividend. If the risky asset is a bond, it is the repayment of the principal plus the interest on the bond. The expected return on an investment in the risky asset is determined by the price we are willing to pay at the beginning of the time period for the right to the risky end-of-period payoff. If P_0, is the price we pay today, our risky return, , is $\widetilde{R}_j = \dfrac{\widetilde{P}_e - P_0}{P_0}.$

The CAPM can be used to determine what the current value of the asset, Po, should be. The CAPM is:

$$\mathrm{E}(\widetilde{R}_j) = (E(\widetilde{P}_e) - P_0)/P_0 \quad (1)$$

$$E(\widetilde{R}_j) = R_0 + (\mu_m - R_0)\frac{\sigma_{mj}}{\sigma_m^2} = R_0 + \frac{(\mu_m - R_0)}{\sigma_m^2}\sigma_{\mathrm{mj}} \quad (2)$$

Which can be rewritten as :

$$E(R_j) = R_f + \lambda \mathrm{COV}(R_j, R_m), \quad \text{where} \quad \lambda = \frac{E(R_m) - R_f}{\mathrm{VAR}(R_m)}.$$

Note that λ can be described as the market price per unit risk. From Eq. (1) and the properties of the mean, we can equate the expected return from Eq. (2) with the expected return in Eq. (3):

$$\frac{E(P_e) - P_0}{P_0} = R_f + \lambda \mathrm{COV}(R_j, R_m).$$

We can now interpret Po as the equilibrium price of the risky asset. Rearranging the above expression, we get:

$$P_0 = \frac{E(P_e)}{1 + R_f + \lambda \mathrm{COV}(R_j, R_m)},$$

So, this is the equation of an equilibrium price of asset j at t=0 (the *risk-adjusted rate of return valuation formula)*

Therefore, the valuation of end of period pay off does not depend on the individual attitudes towards risk. There is a separation of the valuation from the individual attitudes towards risk

3. Applications of the CAPM for corporate policy

The Capital Asset Pricing Model (CAPM) provides a foundational and widely utilized framework for translating systematic risk into a quantifiable expected return. This relationship has several critical applications in finance, particularly in corporate finance and investment management.

The most direct application of the CAPM is to determine the required rate of return on an asset, given its non-diversifiable risk.

- **Security Valuation:** To estimate the intrinsic value of a security (e.g., a stock), future expected cash flows (such as dividends) are discounted to their present value. The CAPM provides the appropriate discount rate that reflects the risk of those cash flows. An asset is considered fairly valued if its market price equals this calculated present value.

- **Capital Budgeting and Project Appraisal:** Corporations use the CAPM to determine the **hurdle rate** or cost of equity capital for evaluating investment projects. By calculating a project-specific beta (reflecting its risk relative to the market), a firm can establish the minimum required return. A project is deemed economically feasible and value-creating only if its expected internal rate of return (IRR) exceeds this CAPM-derived hurdle rate. This ensures that the project compensates shareholders adequately for its systematic risk.

We can use the CAPM to determine the required rate of return on equity. If it is possible to estimate the systematic risk of a company's equity as well as the market rate of return, then E(Rj) is the required rate of return on equity, i.e., the cost of equity for the firm. If we designate the cost of equity as ks, then: $E(R_j) = k_s.$

This is shown in Fig (6.1) As long as all projects have the same risk as the firm, then ks may also be interpreted as the minimum required rate of return on new capital projects.

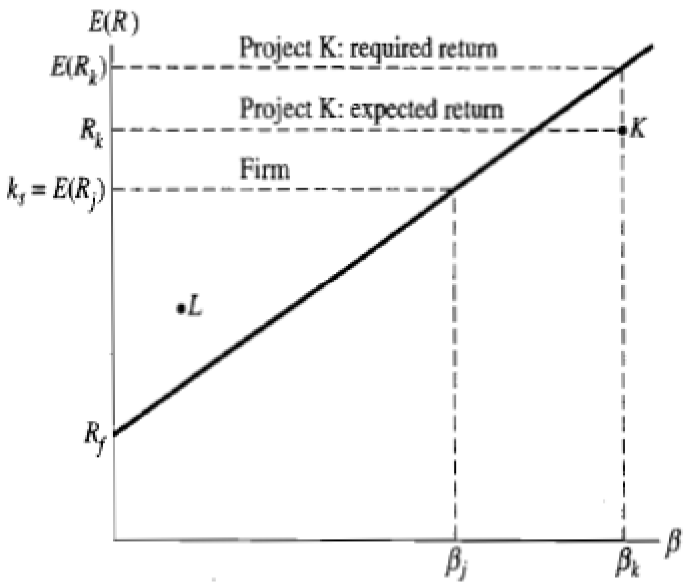


figure 6.1: The cost of equity using the CAPM.

The projected rate of return on project *K, Rk,* is higher than the cost of equity for the firm, E(RJ). But the project also is riskier than the firm because it has greater systematic risk. If the managers of the firm were to demand that it earn the same rate as the firm *[lc, = E(RA,* the project would be accepted since its projected rate of return, *Rk,* is greater than the firm's cost of equity. However, this would be incorrect. The market requires a rate of return, *E(Rk),* for a project with systematic risk of flk, but the project will earn less. Therefore since *Rk* < *E(Rk)* the project is clearly unacceptable. (Is project *L* acceptable? Why?)

Because the CAPM allows decision makers to estimate the required rate of return for projects of different risk, it is an extremely useful concept. Although we have assumed no debt or taxes in the above simple introduction.

4. Calculation and Interpretation of Beta

Beta (β) is a standardized measure of an asset's systematic, or non-diversifiable, risk relative to the broader market. It is a dimensionless statistic that quantifies the sensitivity of an asset's returns to fluctuations in the market portfolio. Specifically, beta indicates the expected change in an asset's return for a one-percent change in the market's return. Mathematically, beta is defined as the ratio of the covariance between the asset's returns and the market's returns to the variance of the market's returns.

$$\beta_i = \frac{\mathrm{Cov}(R_i, R_m)}{\sigma_m^2} = \rho_{i,m} \times \frac{\sigma_i}{\sigma_m}$$

Where:

- βi = Beta of asset i
- Cov(Ri,Rm) = Covariance between the returns of asset i and the market
- σ2m = Variance of the market returns
- ρi,m = Correlation coefficient between the returns of asset i and the market
- σi = Standard deviation of the returns of asset i
- σm = Standard deviation of the market returns

if:

- **β > 1:** The asset is more volatile than the market, amplifying market movements (aggressive stock).
- **β = 1:** The asset's price is expected to move in tandem with the market.
- **0 < β < 1:** The asset is less volatile than the market, dampening market movements (defensive stock).

- **β = 0:** The asset's returns show no correlation with the market movements. A risk-free asset, by definition, has a beta of zero.
- **β < 0:** The asset's returns generally move in the opposite direction to the market (a rare characteristic).

5. Estimation of Beta

Beta is empirically estimated through linear regression analysis, where a time series of historical security returns is regressed against the corresponding returns of a market portfolio. This methodology involves collecting paired data points—for instance, the monthly returns of a stock against a broad market index, such as the S&P 500, over an extended period, typically 60 to 120 months. These data pairs are plotted on a scatter plot, with the market returns on the x-axis and the security returns on the y-axis. A regression line is then fitted to this scatter plot using the least-squares method to derive the line of best fit. The slope of this resulting regression line provides the estimate for beta, quantifying the security's sensitivity to market movements. The y-intercept of the line represents the asset's alpha, indicating historical average excess return, while the dispersion of data points around the line reflects the unsystematic, or idiosyncratic, risk.

The beta of a portfolio βp is calculated as the value-weighted average of the betas of the constituent assets within the portfolio as shown in graph (4.1)

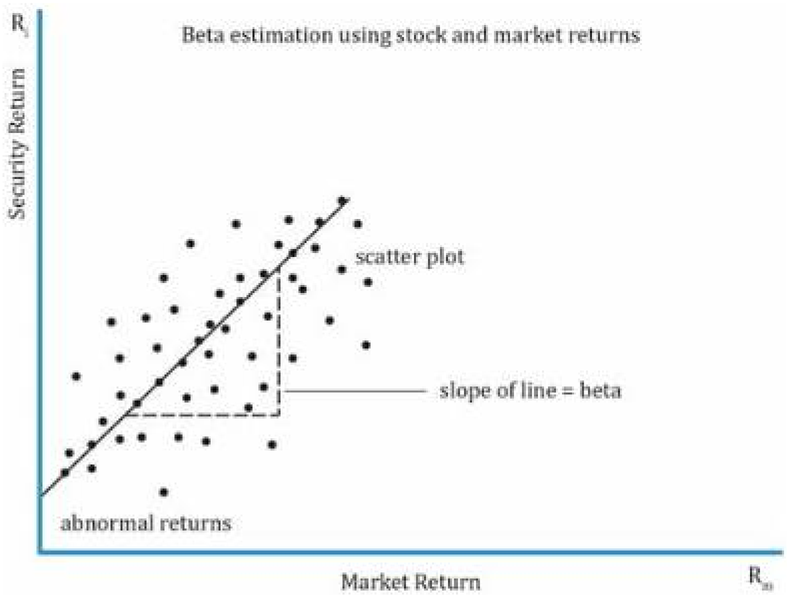


Fig (4.1): Beta estimation

Example:

Given a market return standard deviation of 20%, the beta for each of the following assets is calculated as follows:

The beta for a six-month Treasury bill is zero, as a risk-free asset, by definition, exhibits no covariance with the market portfolio. Similarly, a commodity with a standard deviation of 15% and a correlation of zero with the market also possesses a beta of zero; since its movements are completely uncorrelated with the market, it has no measurable systematic risk. Finally, for a stock with a standard deviation of 25% and a correlation of 0.6 with the market, the beta is calculated using the formula $\beta_i = \rho_{i,m} \times \frac{\sigma_i}{\sigma_m}$. Substituting the given values yields $\beta = 0.6 \times (0.25/0.2) = 0.75$, indicating that the stock is less volatile than the market.

6. Empirical tests on the CAPM

The Capital Asset Pricing Model (CAPM) is a fundamental model in finance that describes the relationship between systematic risk and expected return for assets, particularly stocks. The model comes in two forms: the **ex-ante (theoretical) CAPM** and the **ex-post (empirical) CAPM**.

1. Ex-Ante CAPM (Theoretical Form)

The ex-ante CAPM is based on expectations and states that the **expected return** of an asset *j* is a linear function of its systematic risk (beta, βj):

$\mu j = Rf + \beta j(\mu m - Rf)$
Where:

- μj = Expected return of asset *j*
- Rf = Risk-free rate
- μm = Expected return of the market portfolio
- βj = Systematic risk of asset *j*, defined as $\beta j = Cov(Rj, Rm) / Var(Rm)$

This version is **forward-looking** and relies on investors' expectations.

2. Ex-Post CAPM (Empirical Form)

Since expected returns are not directly observable, empirical tests of the CAPM use **realized (historical) returns** to estimate the relationship. The ex-post CAPM is given by:

$$R_{jt}=R_{ft}+\beta_j(R_{mt}-R_{ft})+\epsilon_{jt}$$

Where:

- R_{jt} = Actual return of asset *j* at time *t*
- R_{ft} = Risk-free rate at time *t*
- R_{mt} = Market return at time *t*
- ϵ_{jt} = Error term (idiosyncratic risk, assumed to have $E[\epsilon_{jt}]=0$)

This is a **time-series regression** that can be estimated using historical data.

To test the CAPM empirically, we often use a **two-step regression approach**, where the second step involves a **cross-sectional regression** to verify whether the model holds. The modified regression model you provided is:

$$R_{jt}'=\gamma_0+\gamma_1\beta_j+\epsilon_{jt}$$

Where:

- $R_{jt}'=R_{jt}-R_{ft}$ (excess return of asset *j* at time *t*)
- β_j = Systematic risk (beta) of asset *j*, estimated from a first-pass time-series regression
- γ_0 = Intercept term (should be zero if CAPM holds)
- γ_1 = Slope term (should equal the market risk premium, $R_{mt}-R_{ft}$, if CAPM holds)

This is a simple linear regression model with the following observations:

a. Hypothesis Test on the Intercept (γ_0)

The CAPM predicts that the intercept (γ_0) should be **zero** in the ex-post model. Thus, we conduct the following hypothesis test:

$$H_0 : \gamma_0 = 0 \quad \text{(CAPM holds)}$$

$$H_1 : \gamma_0 \neq 0 \quad \text{(CAPM does not hold)}$$

Interpretation:

- If we **fail to reject H0**, the data supports the CAPM (intercept is statistically zero).
- If we **reject H0**, the CAPM is misspecified (there may be missing risk factors or mispricing).

b) Beta should be the only factor to explain the rate of return of the risky asset j.

All other factors (such firm size or dividend yield), when included, should have no explanatory power.

c) The relationship should be linear in Beta and only beta.

d) The coefficient of Beta, $\gamma 1$, should be equal to $Rmt-Rft$

e) If estimation is based on a very long period of time, then the rate of return of the market portfolio should be greater than the rate of return of the risk-free asset (because the first is riskier!).

Results of several studies show that:

- The intercept, $\gamma 0$ is significantly different from zero

- The slope, $\gamma 1$ is less than the difference between the rate of the market portfolio (Rm) and the rate of the 90 days treasury bills (Ro). This means that low beta assets have higher rates of return than the high beta assets (again, it is obvious that in real world, economic agents can lend at Rf but they cannot at this rate).

The estimated SML that is provided in the literature **(** (Sharpe & Cooper, 1972)**, cited in** (Aftalion, 2008)**, p. 86) is:**

$$\overline{R}_i = 5.54 + 12.75\,\beta_i \quad \text{With } R_0 = 2\% < 5.54\%$$

This type of estimation has shown (table 2, p. 88 Aftalion) that it may be more interesting to test the model of Black ** (with no risk free asset) rather than the basic one (Sharp, Lintner and Mossin *):

(*): $\mu_j - R_f = \alpha_i + (\mu_m - R_f)\beta_j$

(**): $\mu_j - R_z = (\mu_m - R_z)\beta_j$

In fact, the table shows that applying the initial model explains why the SML slope over estimates the β for companies with small β :

$$\mu_j - R_z = \alpha_i = +(\mu_m - R_z)\beta_j$$

$$\begin{cases}\mu_j - R_f = \alpha_i + (\mu_m - R_f)\beta_j \\ \mu_j - R_z = (\mu_m - R_z)\beta_j\end{cases} \Rightarrow \begin{cases}\mu_j = \alpha_i + R_f + (\mu_m - R_f)\beta_j \\ \mu_j = R_z + (\mu_m - R_z)\beta_j\end{cases}$$

Because Rz is greater than Ro, then we have $\alpha j > 0$ if $\beta j < 0$

- Over long periods of time, $\gamma 1$ is positive.

- Versions of empirical models that use the squared beta or the unsystematic risk terms may be interesting in small samples of periods, but beta as an explanatory factor of the rate of the risky assets dominates all of them.

4. Fama & French Multi-Factor Model

The (Fama & French, 1993) three-factor model was introduced in 1993 to address empirical anomalies that the CAPM could not explain, such as the tendency of small-cap and value stocks to outperform the market.

The three factors are:

- Market Risk Premium (Ri - Rf): The excess return of the market over the risk-free rate, as in CAPM.
- SMB (Small Minus Big): Represents the size effect, capturing the average return of small-cap stocks minus large-cap stocks.
- HML (High Minus Low): Represents the value effect, capturing the average return of high book-to-market (value) stocks minus low book-to-market (growth) stocks.

Mathematically: $R_i - Rf = \alpha + \beta_1 (Ri - Rf) + \beta_2 SMB + \beta_3 HML + \varepsilon$

4.1 Empirical Justification

Empirical studies showed that the CAPM beta alone could not explain the cross-section of average returns. Fama and French demonstrated that small-cap and value stocks earned higher average returns than predicted by CAPM, suggesting omitted risk factors.

Their model improved explanatory power, especially for portfolios sorted by size and value characteristics.

4.2 Model Extensions

Later extensions of the model included additional factors for further improvement:

- - Fama & French Five-Factor Model (2015):Adds two new factors—profitability (RMW) and investment (CMA).
- - RMW (Robust Minus Weak): Difference in returns between firms with robust and weak profitability.
- - CMA (Conservative Minus Aggressive): Difference in returns between firms with conservative and aggressive investment strategies.
- - Carhart Four-Factor Model:Adds a momentum factor (WML – Winners Minus Losers) to capture the tendency of past winners to continue performing well.

5. Extensions of the CAPM model

Nearly all the assumptions underlying the CAPM are violated in reality. This raises an important question: How useful is the model despite these limitations? To answer this, we must address two key aspects: (1) Can the model be extended to relax its unrealistic assumptions without fundamentally altering its core framework? (2) How does the model perform in empirical tests?

a/ No Riskless Asset

First, how will the model change if investors cannot borrow and lend at the riskfree rate? In other words, how is the CAPM affected if there is no risk-free asset that has constant returns in every state of nature? This problem was solved by Black (1972). His argument is illustrated in Fig. (1)

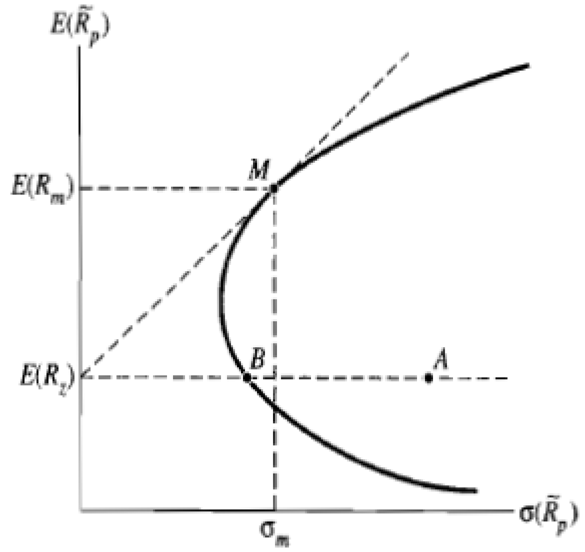


Figure (4.2): The capital market line with no risk-free rate.

In Black's model, the risk-free asset is replaced with a 'zero-beta portfolio': a portfolio uncorrelated with the market and thus bearing no systematic risk (see copeland p159). Portfolio M is identified by the investors as the market portfolio that lies on the efficient set (any efficient portfolio, not just the market portfolio). Portfolios A and B in Fig. 1 are both uncorrelated with the market portfolio M and have the same expected return, E(Rz). However, only one of them, portfolio B, lies on the opportunity set. It is the minimum variance zero-beta portfolio and it is unique. Portfolio A also has zero beta, but it has a higher variance and therefore does not lie on the minimum variance opportunity set.

We can derive the slope of the line $E(R_Z)M$ by forming a portfolio with a% in the market portfolio and (1 — a)% in the minimum variance zero-beta portfolio. The mean and standard deviation of such a portfolio can be written as follows:

$$E(R_p) = aE(R_m) + (1-a)E(R_z),$$

$$\sigma(R_p) = \left[a^2\sigma_m^2 + (1-a)^2\sigma_z^2 + 2a(1-a)r_{zm}\sigma_z\sigma_m\right]^{1/2}.$$

But since the correlation, *r zm,* between the zero-beta portfolio and the market portfolio is zero, the last term drops out. The slope of a line tangent to the efficient set at point M, where 100% of the investor's wealth is invested in the market portfolio, can be found by taking the partial derivatives of the above equations and evaluating them where a = 1. The partial derivative of the mean portfolio return is

$$\frac{\partial E(R_p)}{\partial a} = E(R_m) - E(R_z).$$

And the partial derivative of the standard deviation is:

$$\frac{\partial \sigma(R_p)}{\partial a} = \frac{1}{2}\left[a^2\sigma_m^2 + (1-a)^2\sigma_z^2\right]^{-1/2}\left[2a\sigma_m^2 - 2\sigma_z^2 + 2a\sigma_z^2\right].$$

Taking the ratio of these partials and evaluating where *a = 1*, we obtain the slope of the line *E(RZ)M* in Fig 1:

$$\frac{\partial E(R_p)/\partial a}{\partial \sigma(R_p)/\partial a} = \frac{E(R_m) - E(R_z)}{\sigma_m}.$$

Furthermore, since the line must pass through the point *[E(R.), σ(Rm)],* the interceptof the tangent line must be *E(Rz).* Consequently, the equation of the line must be:

$$E(R_p) = E(R_z) + \left[\frac{E(R_m) - E(R_z)}{\sigma_m}\right]\sigma_p.$$

This is exactly the same as the capital market line except that the expected rate of return on the zero-beta portfolio, *E(Rz)*, has replaced the risk-free rate.

Given the above result, it is not hard to prove that the expected rate of return on any risky asset, whether or not it lies on the efficient set, must be a linear combination of the rate of return on the zero-beta portfolio and the market portfolio. To show this, recall that in equilibrium the slope of a line tangent to a portfolio composed of the market portfolio and any other asset at the point represented by the market portfolio must be equal to

$$\left.\frac{\partial E(R_p)/\partial a}{\partial \sigma(R_p)/\partial a}\right|_{a=0} = \frac{E(R_i) - E(R_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}.$$

If we equate the two definitions of the slope of a line tangent to point *M,* we have*:*

$$\frac{E(R_m) - E(R_z)}{\sigma_m} = \frac{\left[E(R_i) - E(R_m)\right]\sigma_m}{\sigma_{im} - \sigma_m^2}.$$

Solving for the required rate of return on asset *i,* we have: $E(R_i) = (1-\beta_i)E(R_z) + \beta_i E(R_m).$ (1)

$$\beta_i = \sigma_{im}/\sigma_m^2 = \mathrm{COV}(R_i, R_m)/\sigma_m^2.$$

where:

The Equation (1) shows that the expected rate of return on any asset can be written as a linear combination of the expected rate of return of two assets the market portfolio and the unique minimum variance zero-beta portfolio (which is chosen to be uncorrelated with the market portfolio). Interestingly, the weight to be invested in the market portfolio is the beta of the ith asset. If we rearrange (1), we see that it is exactly equal to the CAPM except that the expected rate of return on the zero-beta portfolio has replaced the rate of return on the risk-free asset:

$$E(R_i) = E(R_z) + \left[E(R_m) - E(R_z)\right]\beta_i.$$

The upshot of this proof is that the major results of the CAPM do not require the existence of a pure riskless asset. Beta is still the appropriate measure of systematic risk for an asset, and the linearity of the model still obtains.

b/ The Existence of Nonmarketable Assets

The CAPM assumes that all assets are marketable. But in the real world, it is easy to see that human capital is not marketable (naturally, we are assuming that slavery is not allowed anymore). Therefore, investors cannot diversify away specific risks as stated in the theory of the model. So, for this particular case, investors will end up with no diversifiable assets in their portfolio: their human asset capital! The work of (Mayers, 1972), shows that when investors are constraint to hold no diversifiable assets, they will hold different portfolios of risky assets according to the level of the no diversifiable assets they have. The equilibrium price of risky assets may still be determined independently of the individual's indifference curve; therefore, the price of risk is still objectively determined (the separation principle still hold).

c- The model in continuous time

(Merton, 1973) has introduced two hypotheses:

a/ trading takes place continuously over time,

b/assets returns are lognormal.

It is interesting to note that econometric testing of the CAPM, in the context of Merton assumptions, is consistent with empirical evidence.

d- The existence of heterogeneous expectations and taxes

If investors have different perception of the expected returns, then they will not be able to identify the same efficient set and the same market portfolio. In fact, the concept of market portfolio itself needs to be reanalyzed. However, Lintner (1969) has shown that CAPM can still be used but with expected returns and covariances expressed as complex weighted averages of all investors' expectations. It is important to remark that when we accept the possibility of heterogeneous expectations, then the efficiency of the market portfolio is problematic (Roll's critic).

(Brennan, 1970) when investigating the impact of taxes, has shown that β (beta) is still the appropriate measure of risk level. He concluded however that the CAPM need to be adjust so that the expected return on assets depends on systematic risk and the dividend yield.

It is interesting to note that econometric testing of the CAPM, in the context of Merton assumptions, is consistent with empirical evidence.

6. Alternative Models to the CAPM

Because of the various limitations of the CAPM, several alternative models have been developed to provide a more comprehensive explanation of asset returns. However, these models also have their own limitations. Broadly, they can be divided into theoretical models and practical models.

6.1. Theoretical Models

Theoretical extensions of the CAPM aim to include additional sources of systematic risk beyond the single market factor. One of the most prominent is the Arbitrage Pricing Theory (APT), proposed by (Ross S. A., 1976). The APT expresses the expected return of a portfolio as a linear function of multiple risk factors:

$$\mathbf{E(Rp)=RF+\lambda_1\beta_{p,1}+\lambda_2\beta_{p,2}+\ldots+\lambda_K\beta_{p,k}}$$

where:

- k = number of systematic risk factors,
- λk = risk premium associated with factor k,
- $\beta p,k$ = sensitivity of the portfolio's return to factor k.

The key advantage of the APT is its flexibility — it does not rely on a specific market portfolio as in the CAPM. Instead, it recognizes that asset returns can be influenced by multiple macroeconomic and financial variables such as inflation, interest rates, GDP growth, and exchange rates.

However, the main drawback of the APT is the difficulty in identifying the relevant risk factors and accurately estimating each asset's sensitivity to them. There is no universal agreement on which factors should be included, making practical implementation challenging.

6.2. Practical Models

In practice, several multi-factor models have been developed to address the empirical weaknesses of the CAPM and to improve its predictive power. Among the most influential are the Fama-French factor models.

- The Fama-French Three-Factor Model includes the following factors:
    1. Market risk (as in CAPM),
    2. Size effect (SMB) — the excess return of small firms over large firms,
    3. Value effect (HML) — the excess return of high book-to-market (value) firms over low book-to-market (growth) firms.
- The Fama-French Four-Factor Model extends this framework by adding a momentum factor (MOM), which captures the tendency of assets that performed well in the past to continue performing well in the near future.

These models have been shown to better explain cross-sectional variations in asset returns compared to the traditional CAPM. Nonetheless, they too have limitations:

- They may not apply consistently across all markets or asset classes.
- The factors' effectiveness can change over time as market dynamics evolve.
- There is no guarantee that these models will continue to hold in the future.

Exercises

Exercise 1

You are given the following data for three assets (A, B, C):

Asset	E(Ri)	βi	Rf	E(Rm)
A	12%	0.8	4%	10%
B	15%	1.2	4%	10%
C	18%	1.8	4%	10%

1. Compute the theoretical expected return of each asset according to CAPM.
2. Compare with the actual expected return given in the table.
3. Interpret whether each asset is overvalued, undervalued, or fairly valued.
4. Plot the Security Market Line (SML) and position each asset on the graph.

Exercise 2

You have the following monthly returns over 10 months:

Month	Market Return (Rm)	Stock Return (Ri)
1	0.03	0.05
2	0.01	0.02
3	-0.02	-0.03
4	0.05	0.07
5	-0.01	-0.015
6	0.02	0.03
7	0.04	0.06
8	-0.03	-0.04
9	0.06	0.09
10	0.00	0.01

1. Estimate the beta (β) and alpha (α) of the stock using linear regression.
2. Interpret your results: does the asset have higher or lower systematic risk than the market?
3. Calculate R^2 and interpret its meaning in this context.

Exercise 3

Given the regression results:

$Ri - Rf = 0.005 + 0.9 (Rm - Rf) + \varepsilon i$

where: $t(\alpha) = 2.3$, $t(\beta) = 15.6$, critical $t = 2.0$ at 5% level

1. Test whether $\alpha = 0$ (i.e., CAPM holds).
2. Test whether $\beta = 1$ (the asset has the same risk as the market).
3. Interpret your results economically.

Exercise 4

You are provided with data on 10 portfolios with estimated betas from the first-pass regression.

Portfolio	βi	$(R i - Rf)$
1	0.5	3%
2	0.8	4%
3	1.0	5%
4	1.2	6%
5	1.5	8%
6	1.7	9%
7	2.0	10%
8	2.3	11%
9	2.5	12%
10	2.8	13%

Estimate the Security Market Line (SML) using cross-sectional regression:

$Ri' = \gamma_0 + \gamma_1 \beta i + \varepsilon i$

1. Estimate γ_0 and γ_1.
2. Test whether $\gamma_0 = 0$ and $\gamma_1 = (E(Rm) - Rf)$.

Exercise 5

You are given the following factor premiums and stock sensitivities:

Factor	Value	
Rm - Rf	6%	
SMB	2%	
HML	3%	
βM	**βSMB**	**βHML**
1.2	0.4	-0.3

1. Compute the expected return of the asset using the Fama-French model: $E(Ri) - Rf = \beta M(Rm - Rf) + \beta SMB(SMB) + \beta HML(HML)$
2. Compare it with the CAPM-predicted return.
3. Explain why the difference arises.

Chapter 5: Applications and Extensions of the CAPM model

In the realm of modern finance, the accurate pricing of financial assets and the evaluation of investment performance are fundamental to portfolio management, asset allocation, and financial decision-making. The **Capital Asset Pricing Model (CAPM)** has long served as a cornerstone in asset pricing theory, offering a simplified framework to understand the trade-off between risk and return. Despite its theoretical elegance, CAPM has faced increasing scrutiny due to its restrictive assumptions and empirical limitations.

This has led to the development of extended models and new approaches that better reflect the complexity of financial markets. Among them, the **Fama and French multi-factor models** and the **Arbitrage Pricing Theory (APT)** have provided more flexible and empirically robust alternatives to single-factor models. These frameworks incorporate multiple sources of systematic risk and better capture the cross-sectional variation in asset returns.

Simultaneously, the evaluation of mutual fund performance has become increasingly sophisticated. Investors and academics alike rely on risk-adjusted performance measures such as the Sharpe ratio**,** Treynor ratio, and Jensen's alpha to assess whether active fund managers deliver true value beyond market benchmarks. Additionally, as empirical evidence highlights asymmetries in return distributions**,** three-moment portfolio models, which account for skewness, have emerged as more realistic tools for understanding investor preferences.

This chapter aims to explore the empirical validation and extensions of CAPM, critically compare it with alternative asset pricing models, and examine their applications in performance measurement and portfolio theory. Through this, we aim to highlight both the theoretical underpinnings and practical implications of modern financial models in an evolving market environment.

1. Capital Asset Pricing Model: Applications

The Capital Asset Pricing Model (CAPM) serves as a fundamental framework in financial economics with significant practical utility across various domains. Its primary applications include:

- **Security Valuation**

  ✓ Determining the appropriate risk-adjusted discount rate for valuing equities

  ✓ Calculating intrinsic value by discounting expected future cash flows

- ✓ Identifying mispriced securities by comparing CAPM-derived expected returns with actual returns

- **Capital Budgeting and Corporate Finance**

  - ✓ Establishing hurdle rates for project evaluation
  - ✓ Calculating the cost of equity capital for WACC computation
  - ✓ Making capital structure decisions based on risk-adjusted return requirements

- **Performance Evaluation**

  - ✓ Assessing portfolio manager performance using Jensen's alpha
  - ✓ Measuring risk-adjusted returns through the Treynor ratio
  - ✓ Comparing actual returns against CAPM-derived expected returns

- **Regulatory Applications**

  - ✓ Setting fair rates of return in regulated industries
  - ✓ Cost of capital estimation for utility pricing and insurance ratemaking

- **Expected Return Estimation**

  - ✓ Providing forward-looking return expectations for asset allocation
  - ✓ Forming capital market assumptions for long-term financial planning

The model's enduring relevance stems from its ability to provide a theoretically sound, market-based estimate of required returns that explicitly accounts for systematic risk exposure.

2. Applications of the CAPM in Portfolio Construction

2.1 Security Characteristic Line

The Security Characteristic Line (SCL) represents the relationship between the excess return of a security and the excess return of the market portfolio. It is typically plotted with the security’s excess return over the risk-free rate on the y-axis, and the market’s excess return on the x-axis.

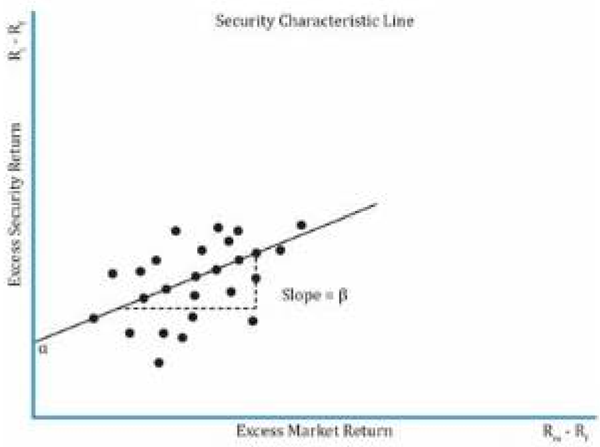


Fig (5.1): Security Characteristic Line

The SCL provides a visual and analytical method for understanding how a specific security's returns move in relation to market movements. The intercept of the line (αi) measures the portion of the security's return that is independent of the market—often referred to as Jensen's alpha—while the slope of the line (βi) represents the sensitivity of the security's return to changes in the market return.

Mathematically, the Security Characteristic Line can be expressed as:

$$R_i - R_f = \alpha_i + \beta_i (R_m - R_f)$$

where:

$R_i - R_f$ = excess return of the individual security

$R_m - R_f$ = excess return of the market portfolio

α_i = Jensen's alpha (the security's abnormal return, unexplained by market movements)
β_i = beta coefficient (the sensitivity of the security's return to the market return)

This relationship is typically estimated using regression analysis, where the dependent variable is the excess return of the security and the independent variable is the excess return of the market. The estimated regression coefficients provide the values of alpha and beta.

Interpretation of the SCL:

- A positive alpha ($\alpha i > 0$) indicates that the security outperforms its expected return according to the CAPM.
- A negative alpha ($\alpha i < 0$) indicates underperformance relative to the CAPM benchmark.
- The beta (βi) measures the systematic risk of the security. A beta greater than 1 implies higher sensitivity to market movements, while a beta less than 1 implies lower sensitivity.

The SCL is therefore a valuable analytical tool for portfolio managers, as it helps to decompose total returns into market-related and security-specific components, supporting more informed investment and diversification decisions.

2.2 Security Selection

In the Capital Asset Pricing Model (CAPM), we assume investors have homogeneous expectations and value all assets identically, leading to the conclusion that all investors hold the same optimal risky portfolio: the market portfolio. However, in practice, this assumption rarely holds true as investors often have differing expectations and valuation methods.

The Security Market Line (SML) serves as a practical tool for security selection. Investors can assess individual securities by plotting their expected returns against their respective beta coefficients on the SML graph (5.2):

- A security plotting **above the SML** is considered **undervalued**, as it offers higher expected return than warranted by its systematic risk level.
- A security plotting **below the SML** is considered **overvalued**, as it provides lower expected return than justified by its risk.
- A security plotting **directly on the SML** is considered **fairly priced**, as its return adequately compensates for its systematic risk.

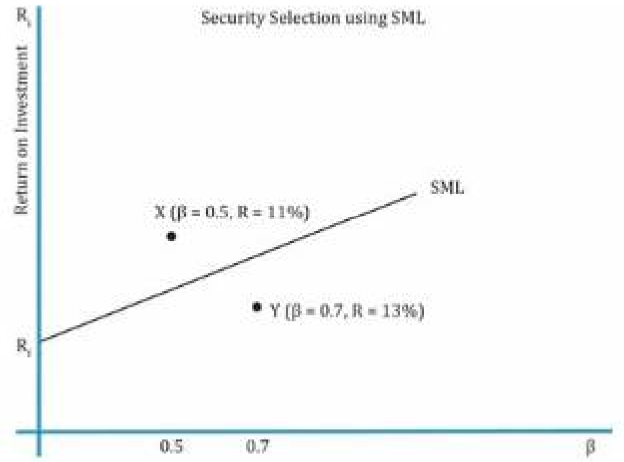


Fig (5.2): Security Selection using SML

As illustrated in the exhibit, Security X (β = 0.5, R = 11%) appears above the SML, indicating it is undervalued. At this risk level (β = 0.5), Security X offers a higher return than a security with similar risk positioned on the SML. Conversely, Security Y (β = 0.7, R = 13%) plots below the SML, suggesting it is overvalued. At this risk level (β = 0.7), an alternative security on the SML would provide a superior risk-adjusted return compared to Security Y.

3. Performance Evaluation of Mutual Funds

To assess whether fund managers generate abnormal returns (also called alpha) after accounting for risk.

3.1.Key Performance Measures:

Measure	Formula	Interpretation
Sharpe Ratio	$R_p - R_f / \sigma_p$	Excess return per unit of total risk (volatility)
Treynor Ratio	$R_p - R_f / \beta_p$	Excess return per unit of market risk (systematic risk)
Jensen's Alpha	$\alpha = R_p - [R_f + \beta_p (R_m - R_f)]$	Measures how much return a fund earned above what CAPM predicts
M-Squared (M^2)	$M^2 = \frac{(R_p - R_f)\sigma_m}{\sigma_p} - (R_m - R_f)$	Measures Risk-Adjusted Performance.

A good mutual fund doesn't just outperform the market; it adds value adjusted for the risk it takes.

3.1.1 The Sharpe Ratio (SR) measures the excess return earned by a portfolio per unit of its total risk. It is defined as the difference between the portfolio's return and the risk-free rate, divided by the standard deviation of the portfolio's returns. A higher Sharpe Ratio indicates a superior risk-adjusted performance, all else being equal. Graphically, it represents the slope of the Capital Allocation Line (CAL) and is commonly referred to as the reward-to-variability ratio.

The ratio can be calculated from both ex-ante (forward-looking) and ex-post (historical) perspectives:

- **Ex-ante Sharpe Ratio**= $E(Rp)-Rf/\sigma p$ *:*This formulation is used to evaluate the *expected* risk-adjusted return of a portfolio based on projected returns and risk.
- **Ex-post Sharpe Ratio**=$\breve{Rp}$-$\breve{Rf}$/$\breve{\sigma p}$: This formulation is used to evaluate *historical* risk-adjusted performance, where $\breve{Rp}$ and $\breve{Rf}$ are the historical average returns of the portfolio and the risk-free asset, respectively, and $\breve{\sigma p}$ is the historical standard deviation of portfolio returns.

a. **Advantages of the Sharpe Ratio**

- **Simplicity:** It can be easily calculated using readily available market data.
- **Interpretability:** The result is intuitive and straightforward to understand.
- **Prevalence:** It is the most widely recognized and utilized performance appraisal measure in finance.

b. Limitations of the Sharpe Ratio

- **Reliance on Total Risk:** The ratio uses total risk (standard deviation) as its denominator, which includes both systematic and unsystematic risk. This can penalize portfolios that are not fully diversified, even if the unsystematic risk is acceptable to the investor.
- **Relative, Not Absolute, Measure:** The ratio's value in isolation is often uninformative. A portfolio with a Sharpe Ratio of 0.7 provides no inherent benchmark; it must be compared to the ratio of another portfolio (e.g., one with 0.9) or a relevant market index to determine relative performance superiority.

3.1.2 The Treynor Ratio: is a performance metric for a portfolio that measures the excess return earned per unit of **systematic risk** (beta).

- **Ex-ante:** $TR = \frac{E(R_p) - R_f}{\beta_p}$
- **Ex-post :** $\widehat{TR} = \frac{\hat{R}_p - \hat{R}_f}{\hat{\beta}_p}$

- The numerator (excess return) must be positive for the ratio to be meaningful.
- It is not suitable for assets with a negative beta.

❖ **Limitations:**

1. **Lacks an Absolute Benchmark:** While it can rank portfolios (a higher ratio is better), the ratio itself does not indicate if the performance is better than the market portfolio.
2. **Does Not Quantify the Performance Difference:** It shows the order of performance but not the magnitude. For example, a portfolio with a ratio of 0.7 is better than one with 0.6, but we don't know by how much.

3.1.3. M-Squared (M^2): The M-Squared measure (also known as the Modigliani–Modigliani measure or risk-adjusted performance measure, RAP) evaluates a portfolio's performance adjusted for total risk.

It transforms the Sharpe Ratio into a percentage return that can be directly compared to the market's performance. If the M^2 of a portfolio is greater than the market return, it means that the portfolio provides a higher risk-adjusted return than the market.

If $\mathbf{M^2 = 0}$, the portfolio's risk-adjusted return equals the market's.

If $\mathbf{M^2 < 0}$, the portfolio underperforms the market on a risk-adjusted basis.

Ex-ante form: $$M^2 = [E(R_p) - R_f]\frac{\sigma_m}{\sigma_p} + R_f = SR \times \sigma_m + R_f$$

Ex-post form: $$\widehat{M^2} = [\hat{R}_p - R_f]\frac{\hat{\sigma}_m}{\hat{\sigma}_p} + R_f = \widehat{SR} \times \hat{\sigma}_m + R_f$$

M^2 expresses the risk-adjusted performance of a portfolio in percentage terms — that is, the return the portfolio would have achieved if it had the same total risk as the market**.**

$$M^2 - R_f = SR \times \sigma_m$$

Thus, the M² value allows a direct comparison with the market's performance.

a. Advantages

- Expressed in percentage units, M² is easy to interpret and communicate.
- Allows a direct comparison between portfolios with different levels of total risk.
- Provides an intuitive complement to the Sharpe Ratio.

b. Limitations

- M² relies on total risk (standard deviation) rather than systematic risk (beta).
- Therefore, it might not be appropriate for assessing portfolios that are well-diversified or for performance attribution relative to systematic factors.

3.1.4. Jensen's Alpha is a performance metric that measures a portfolio's risk-adjusted return by calculating the difference between its actual return and the return predicted by the Capital Asset Pricing Model (CAPM).

- It is the intercept (alpha) in a regression of the portfolio's excess returns against the market's excess returns, with the portfolio's beta as the slope.

a. Interpretation:

- **Alpha > 0:** The portfolio has outperformed the market on a risk-adjusted basis.
- **Alpha = 0:** The portfolio's performance matched the market's expectations.
- **Alpha < 0:** The portfolio underperformed the market on a risk-adjusted basis.

b. Application and Comparison:

- Like the Treynor Ratio, Jensen's Alpha is based on systematic risk (beta)**.**
- It is considered theoretically superior to measures like M-Squared for ranking well-diversified portfolios and their managers.
- **Instructor's Note:** Sharpe Ratio and M-Squared use total risk, making them suitable for non-fully diversified portfolios. Treynor Ratio and Jensen's Alpha use beta risk, making them suitable for well-diversified portfolios.

Sharpe Ratio and M-Squared use total risk, making them suitable for non-fully diversified portfolios. Treynor Ratio and Jensen's Alpha use beta risk, making them suitable for well-diversified portfolios.

4. Skewness and Three-Moment Portfolio Models

Traditional portfolio theory, based on Harry Markowitz's (1952) mean–variance framework, assumes that investors make decisions by considering only two parameters**:**

- the expected return of the portfolio (mean), and
- Its risk, measured by the variance (or standard deviation) of returns.

However, empirical evidence shows that asset returns are often not symmetrically distributed, meaning they are skewed (asymmetric) or exhibit fat tails (kurtosis).

This asymmetry leads to deviations from the normal distribution assumption underlying the mean–variance model.

To better represent investor preferences under non-normal return distributions, three-parameter portfolio models have been developed, which take into account skewness (asymmetry) in addition to mean and variance.

By accounting for skewness, three-moment models provide a more realistic and flexible framework for portfolio construction. They also help explain certain behavioral patterns observed in financial markets, such as the persistent demand for lottery-like assets and the popularity of speculative strategies, which cannot be justified under the standard mean-variance paradigm.

3.1. Understanding Asymmetry in Financial Returns

- Asymmetry (or skewness) measures the degree of distortion from a symmetrical bell-shaped (normal) distribution.
- In finance, it describes whether the distribution of returns has a longer tail on one side**.**

4.1.1. Types of Skewness

a) Positive (right) skewness:

  - The right tail is longer; a few large positive returns pull the average upward.

- Investors tend to prefer positively skewed returns because they imply potential for large gains.
- Example: options or venture capital returns.
-

b) **Negative (left) skewness**:

- The left tail is longer; extreme negative returns are more frequent.
- Investors dislike negative skewness because it implies a higher probability of large losses.
- Example: insurance companies, short positions, or carry trade strategies.

4.1.2. Empirical Observation

Financial asset returns are rarely normal; they tend to show:

- Fat tails (high kurtosis): extreme events occur more often.
- Negative skewness: crashes or losses tend to be larger than gains.

4.2. Limitations of the Mean–Variance Model

- The mean–variance framework assumes: $Rp \sim N(\mu p, \sigma^2 p)$. This implies symmetric returns and no extreme outliers.
- Traditional models assume normal distribution of returns (i.e., symmetric).
- However, in reality, financial returns often show skewness (asymmetry).

However:

- Real-world data often violate normality.
- Two portfolios with identical mean and variance can have very different skewness, leading to different levels of attractiveness for investors.

Thus, relying only on expected return and variance can produce suboptimal portfolio choices in the presence of asymmetry.

5. The Three-Parameter Portfolio Selection Model

The three-parameter model extends the Markowitz approach by incorporating skewness (S) as a third dimension of investor preference.

The investor's utility function can be expressed as: $U=f(\mu,\sigma^2,S)$

Where:

- μ: Expected return
- σ^2: Variance (risk)
- S: Skewness (asymmetry)

Investors may care about not just mean (1st moment) and variance (2nd moment), but also skewness (3rd moment):

$$U = E(R) - \frac{1}{2}A\sigma^2 + \frac{1}{6}\gamma S$$

Where:

- $A>0$: risk-aversion coefficient
- $\gamma >0$: skewness preference coefficient
- E(R): Expected return
- σ^2: Variance
- S: Skewness

While classical mean-variance models focus solely on expected return and risk (variance), they overlook higher moments such as skewness, which captures the asymmetry of return distributions. Incorporating skewness into portfolio theory offers deeper insight into investor behavior and asset allocation.

a. **Investor Preference for Positive Skewness**
   Investors generally favor positively skewed assets—those with frequent small losses and occasional large gains—because they offer potential for high upside while limiting downside risk.
b. **Impact on Portfolio Composition**
   When skewness is considered, optimal portfolios shift toward assets with favorable asymmetry, leading to allocations that differ from those generated by mean-variance optimization. This reflects a more realistic approach to investor preferences under uncertainty.

c. **Explanation for Demand for Lottery-like Assets**
   The preference for positive skewness helps explain why some investors are drawn to lottery-type investments or speculative strategies (e.g., certain hedge funds), despite their lower expected returns or higher volatility under traditional models.

5.1. Implications for Portfolio Selection

In the three-parameter setting, investors choose portfolios that:

1. Maximize expected return (μp),
2. Minimize variance ($\sigma^2 p$),
3. Maximize skewness (Sp).

Thus, portfolios are evaluated in a three-dimensional space rather than the traditional mean–variance frontier.

5.2. Empirical and Practical Applications

- **Option portfolios**: tend to display strong positive or negative skewness; hence, incorporating skewness helps evaluate tail-risk exposure.
- **Hedge funds and private equity**: often exhibit non-normal, skewed return distributions.
- **Performance evaluation**: using three moments (mean, variance, skewness) provides a more complete picture of risk–return trade-offs.

5.3. Advantages of the Three-Parameter Model

1. **Realism**: Accounts for the non-normal distribution of returns observed in financial markets.
2. **Investor Behavior**: Reflects the fact that investors prefer positively skewed outcomes.
3. **Improved Portfolio Diversification**: Encourages inclusion of assets that contribute positively to portfolio skewness.
4. **Better Risk Management**: Incorporates downside risk perception more effectively than variance alone.

5.4. Limitations

1. **Complexity**: Estimating co-skewness and higher-order moments is mathematically demanding.
2. **Data sensitivity**: Skewness estimates are unstable for small samples.
3. **Interpretation difficulty**: The efficient surface in three dimensions is harder to visualize and optimize.
4. **Non-linear optimization**: Requires numerical methods rather than simple analytical solutions.

5.5. Extensions

Recent research expands on this idea by incorporating kurtosis (fat tails), leading to four-moment portfolio models: $U=f(\mu,\sigma^2,S,K)$
where K is kurtosis. Such models allow even finer adjustment for risk associated with extreme events.

The three-parameter portfolio selection model represents a significant advancement over the traditional mean–variance approach.

By integrating asymmetry (skewness) into investment decisions, it reflects more accurately how investors behave in real markets, where returns are rarely symmetric.

It provides a richer, more realistic framework for asset allocation, especially for portfolios exposed to non-normal risks, such as derivatives, hedge funds, and emerging market investments.

6. Arbitrage Pricing Theory (APT)

The Arbitrage Pricing Theory (APT), introduced by (Ross, 1976) (Markowitz, 1952), represents a major advancement in asset pricing theory. Unlike the Capital Asset Pricing Model (CAPM), which relies on a single market factor and strong assumptions about market equilibrium, APT provides a more flexible, multi-factor framework for explaining asset returns.

6.1 Definition of Arbitrage

Arbitrage refers to the simultaneous purchase and sale of equivalent securities in different markets to exploit price discrepancies and earn risk-free profits. In equilibrium, arbitrage opportunities disappear as market forces adjust prices.

APT is built on the idea that if securities are mispriced relative to their risk exposure, arbitrageurs will trade until equilibrium is restored.

6.2. The APT Model

A second pivotal model, Arbitrage Pricing Theory (APT)**,** introduced by Ross [1976], shares the CAPM's equilibrium pricing framework but generalizes it. Instead of a single market factor, APT expresses asset returns as a linear combination of multiple systematic risk factors. While APT encompasses the CAPM as a special case, its broader structure allows for a more flexible analysis of asset pricing.

APT is based on the idea that asset prices should reflect a linear combination of various macroeconomic and firm-specific factors, such as interest rates, inflation, GDP growth, or industrial production. The model assumes that if assets are mispriced relative to their factor exposures, arbitrage opportunities will emerge, and rational investors will act to eliminate them, restoring equilibrium.

The APT equation for the expected return of asset *i* is:

$$E(R_i) = R_f + b_{i1}\lambda_1 + b_{i2}\lambda_2 + \ldots + b_{in}\lambda_n$$

- Rf = Risk-free rate
- βik = Sensitivity (factor loading) of asset *i* to factor *j*
- λ_k= Risk premium associated with factor *j* (i.e., $E(Fj)-Rf$)

APT does not specify which factors to use—they can be macroeconomic, fundamental, or statistical. Common empirically identified factors include:

1. **Market Risk** (similar to CAPM)
2. **Inflation Rate**
3. **Interest Rate Changes**

4. **Industrial Production/GDP Growth**
5. **Term Structure of Interest Rates** (Yield Curve Slope)
6. **Credit Spreads** (Corporate Bond Yields - Treasury Yields)
7. **Commodity Prices** (Oil, Gold, etc.)

6.3. Assumptions

a. **No Arbitrage Condition**: If two portfolios have identical risk, they must offer the same return; otherwise, arbitrageurs will exploit the mispricing until equilibrium is restored.
b. **Factor Model of Returns**: Asset returns are linearly influenced by a set of common risk factors.
c. **Large Number of Assets**: The market has enough securities for diversification to eliminate idiosyncratic (firm-specific) risk.
d. **No Market Imperfections**: No taxes, transaction costs, or restrictions on short-selling (in the basic model).

6.4. Economic Interpretation

Each factor F_k represents a systematic source of uncertainty that affects all assets in the economy. For example:

Factor	Economic Interpretation	Possible Proxy
F1	Interest rate risk	Treasury yield changes
F2	Inflation risk	CPI or inflation surprises
F3	Industrial production	GDP growth rate
F4	Exchange rate risk	Currency index

The coefficient b_{ik} measures how sensitive the asset *i* is to changes in each factor. The risk premium λ_k represents the additional expected return for bearing one unit of risk associated with factor *k*.

6.5. Derivation of the APT Relationship

APT is derived from the no-arbitrage condition. If two well-diversified portfolios, *A* and *B*, have identical factor sensitivities, then their expected returns must be equal:

$$\mathbf{E(RA)-Rf = E(RB)-Rf}$$

Otherwise, investors could construct a zero-investment portfolio (long one, short the other) and earn a risk-free arbitrage profit.
Therefore, equilibrium requires:

$$E(Ri) - Rf = b_{i1}\lambda_1 + b_{i2}\lambda_2 + ... + b_{in}\lambda_n$$

This linear relationship between expected return and factor sensitivities is the APT pricing equation.

6.6. APT vs. CAPM

Feature	**APT**	**CAPM**
Number of Factors	Multiple (flexible)	Single (Market Risk)
Assumptions	No arbitrage, factor structure	Efficient markets, mean-variance investors
Dependence on Market Portfolio	No (does not require market portfolio)	Yes (relies on market beta)
Practical Use	More general but harder to implement	Simpler but restrictive

6.7. Empirical Testing of APT:

APT is tested using:

- **Factor Analysis**: Statistically extracts common factors from return data.
- **Macroeconomic Models**: Uses observable economic variables ((Chen, 1986)
- **Fama-French Models**: Extend APT with size (SMB) and value (HML) factors.

APT is a powerful alternative to CAPM, offering a more nuanced view of risk and return. However, its reliance on factor identification makes it more complex to implement in practice. Modern asset pricing models (e.g., Fama-French 5-factor) build upon APT's multi-factor framework.
The APT Commonly used factors include:

- Term structure of interest rates
- Inflation expectations
- Industrial production

- Market index returns
- Energy prices

The model can be estimated using multiple regression analysis:

$$Ri - Rf = b_{i1}F_1 + b_{i2}F_2 + ... + b_{in}F_n + \epsilon i$$

6.8. Advantages and Limitations

a. Advantages

- More realistic than CAPM (multiple sources of risk).
- Flexible and empirically testable.
- Fewer restrictive assumptions about investor behavior.

b. Limitations

- Does not identify the relevant risk factors.
- Requires large data sets and complex estimation.
- Assumes perfect market conditions and no arbitrage.

The Arbitrage Pricing Theory (APT) provides a powerful and flexible framework for explaining asset returns. By recognizing that returns are influenced by multiple systematic factors, it enhances our understanding of risk–return relationships in financial markets. APT complements and extends CAPM and serves as a foundation for multi-factor models such as the Fama–French three-factor model and Carhart four-factor model used in modern portfolio management.

Example

Suppose an investor is evaluating a stock influenced by two factors:

- F1: Interest rate changes ($\lambda 1=2\%$)
- F2: Inflation rate ($\lambda 2= 4\%$)

Given: $Rf= 3\%$, $bi_1= 1.2$, and $bi_2= 0.5$.

Then:

$E(R_i)=3\%+(1.2)(2\%)+(0.5)(4\%)=3\%+2.4\%+2.0\%=7.4\%$

The Return for the asset is 7.4%.

Exercices

Exercise 1

A stock has an expected return of 13%, while the risk-free rate is 4%, and the expected market return is 10%.

1. Calculate the theoretical return of the stock using the CAPM model if its beta is 1.2.
2. Determine whether the stock is overvalued, undervalued, or fairly priced.
3. Explain how such analysis would guide an investment decision.

Exercise 2:

A firm is evaluating a new project. The expected cash flow is 1,200,000 per year for 5 years. The project's beta is 1.5, the market return is 9%, and the risk-free rate is 3%.

1. Compute the cost of equity using the CAPM.
2. Determine the present value of cash flows if this rate is used as the discount rate.
3. Discuss whether the firm should accept the project based on the calculated net present value (NPV).

Exercise 3:

Two mutual funds, A and B, have the following characteristics:

Fund	Average Return Rp	Standard Deviation σp	Beta βp	Risk-free Rate Rf
A	14%	8%	1.1	4%
B	12%	5%	0.8	4%

1. Calculate the Sharpe and Treynor ratios for both funds.
2. Interpret which fund performed better under each measure.
3. Explain why the rankings may differ between the two ratios.

Exercise 4

A mutual fund has the following data:

Average return = 15%, Beta = 1.2, Market return = 10%, Risk-free rate = 3%, Portfolio standard deviation = 9%, Market standard deviation = 6%.

1. Compute Jensen's Alpha for the portfolio.
2. Compute the M^2 measure and compare it with the market return.
3. Interpret the meaning of these results in terms of fund manager skill and risk-adjusted performance.

Exercise 5:

You are given three assets with the following characteristics:

Asset	Expected Return (μ)	Variance (σ^2)	Skewness (S)
X	8%	0.04	0.5
Y	10%	0.06	-0.3
Z	12%	0.08	0.8

An investor has a utility function $U = \mu - 0.5A\sigma^2 + 0.1\gamma S$, where $A = 3$ and $\gamma = 2$.

1. Compute the utility of each asset.
2. Identify which asset the investor would prefer under this utility framework.
3. Explain the impact of skewness preference on portfolio selection.

Exercise 6

A researcher estimates the following APT model:

$E(Ri) = Rf + bi1\ \lambda1 + bi2\ \lambda2$

with two factors:

- Factor 1: market risk ($\lambda1 = 6\%$)
- Factor 2: interest rate sensitivity ($\lambda2 = 2\%$)

The following factor loadings are observed:

Asset	b1	b2
G	1.1	0.3
H	0.8	0.7

1. Compute the expected return for each asset under APT.

2. Compare the result with the single-factor CAPM result (assuming β = b1).
3. Interpret differences and discuss when APT provides a better description of returns.

Exercise 7

As a financial analyst, you are comparing the explanatory power of the CAPM and the Arbitrage Pricing Theory (APT) for a particular stock.

You are provided with the following data on expected returns and risk factors for three securities (A, B, C), as well as information about two risk factors identified in the market (inflation changes and GDP growth surprises).

Security	**β (Market Beta)**	**Sensitivity to Inflation (β_1)**	**Sensitivity to GDP (β_2)**	**Expected Return (E[Ri])**	**Risk-Free Rate (Rf)**
A	1.2	0.6	1.4	12.8%	4%
B	0.9	0.3	0.9	10.0%	4%
C	1.5	0.8	1.6	14.5%	4%

Additional information:

- The expected market risk premium (E[Rm] – Rf) = 6%
- The risk premiums for inflation and GDP factors are respectively:
  - λ_1 = 1.5%
  - λ_2 = 2.0%

Questions

1. Use the CAPM formula to calculate the theoretical expected return for each security (A, B, C).

2. Compare your CAPM results with the actual expected returns given in the table. Which securities appear overvalued or undervalued according to the CAPM?

3. Use the two-factor APT model to compute the expected return for each security under the APT.

4. Compare your APT expected returns with the actual expected returns. Which model (CAPM or APT) better explains the observed returns?

5. Discuss the implications of your results for each model's predictive power.

6. What do your findings suggest about the limitations of CAPM and the advantages of multi-factor models such as the APT?

7. Explain how the Fama-French three-factor model attempts to improve on both CAPM and APT.

8. Identify the three factors and discuss why they capture additional sources of risk.

Conclusion

The study of portfolio theory and asset pricing models provides essential insights into how investors make rational decisions under uncertainty and how financial assets are valued in both theoretical and practical contexts. Beginning with the mean-variance approach, this framework established a formal definition of the portfolio selection problem, linking risk and return in a quantifiable manner. The normative models introduced key tools such as the efficient frontier, minimum variance portfolios, and the risk-return trade-off, enabling investors to construct portfolios aligned with their risk preferences.

The inclusion of a risk-free asset, as demonstrated through the Separation Theorem and the Capital Market Line (CML), further refined these models, highlighting the role of market equilibrium and the benefits of diversification. In this context, models like the diagonal model and market model offered simplified yet powerful ways to analyze portfolio behavior, even under constraints such as short-sale restrictions.

The development of the Capital Asset Pricing Model (CAPM) marked a major milestone, providing an elegant equilibrium-based approach to pricing risky assets using a single risk factor: market beta. Its graphical expression, the Security Market Line (SML), offered a clear benchmark for expected returns. However, empirical testing has revealed that CAPM, while foundational, does not fully capture the complexities of real markets. This has led to the emergence of extended frameworks such as the Fama and French multi-factor models, which integrate size and value factors to improve explanatory power, and the Arbitrage Pricing Theory (APT), which allows for a broader and more flexible multi-factor structure without relying on a market portfolio.

Moreover, recent developments have expanded the application of these models beyond pricing, into performance evaluation of mutual funds, incorporating measures like Sharpe ratio, Jensen’s alpha, and Information ratio. Recognizing the importance of asymmetry in return distributions, three-moment models have also been introduced to reflect investor preferences for skewness, explaining demand for lottery-like assets and certain hedge fund strategies.

In conclusion, the evolution from basic portfolio selection to multi-factor asset pricing models and performance metrics reflects the dynamic and multidimensional nature of financial markets. Mastery of these concepts equips financial professionals and researchers with the analytical tools needed to navigate investment decisions, manage risk, and contribute to the ongoing development of modern finance theory.

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